# Population Dynamics and Stochastic Particle Systems

by


**Watthanan Jatuviriyapornchai**


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# Declarations

This work has been composed by myself and has not been submitted for any other degree or professional qualification.

- Chapter 2 introduces standard notations and well-known results from recent literature.
- Chapter 3 contains original work jointly conducted with Stefan Grosskinsky. A paper based on this work has been submitted to Stochastic Processes and their Applications and is under review [61].
- Chapter 4 contains original work jointly conducted with Stefan Grosskinsky and has been published in [60].
- Chapter 5 contains original work jointly conducted with Stefan Grosskinsky and will be submitted for publication.

# Abstract

Condensation is a special class of phase transition which has been observed throughout the natural and social sciences. The understanding of dynamics towards condensation on a mathematically rigorous level is currently a major research topic. Starting the system from homogeneous initial conditions, the time evolution of the condensed phase often exhibits an interesting coarsening phenomenon of mass transport between cluster sites. In this thesis, we study the coarsening dynamics in several condensing stochastic particle systems.

First, we consider the single site dynamics in general stochastic particle systems of misanthrope type with bounded rates on a complete graph. In the limit of diverging system size, we establish convergence to a Markovian non-linear birth death chain, described by a mean-field equation also known from exchange-driven growth processes. Conservation of mass in the particle system leads to conservation of the first moment for the limiting dynamics, and to non-uniqueness of stationary measures. The proof is based on a coupling to branching processes via the graphical construction and establishing uniqueness of the solution for the limit dynamics. As particularly interesting examples we discuss the dynamics of two models that exhibit a condensation transition and their connection to exchange-driven growth processes.

The first model is the zero-range process with bounded jump rates. It is well known that zero-range processes with decreasing jump rates exhibit a condensation transition under certain conditions. The mean-field limit of the single site dynamics leads to a non-linear birth death chain describing the coarsening behaviour. We introduce a size-biased version of the single site process, which provides an effective tool to analyse the dynamics of the condensed phase without finite size effects. The second model is the inclusion process, which has unbounded jump rates and also exhibits the condensation phenomenon. However, in this case, the mean-field equation is derived differently, and the single site process is in the form of a standard birth death chain. In addition to the site and size-biased processes, we derive some exact results on the system through duality. We compute the time dependent covariance using the self-duality of inclusion processes and a two-particle dual process. Our results are based on exact computations and are corroborated by detailed simulation data, which contribute to a rigorous understanding of the approach to stationarity in the thermodynamic limit of diverging system size and particle number.

# Notations

| | |
|---|---|
| $\mathbb{N}_0 = \{0, 1, 2, ...\}$ | nonnegative integer numbers including 0 |
| $\mathbb{N} = \{1, 2, 3, ...\}$ | positive integer numbers |
| $\delta_{.,.}$ | Kronecker delta function |
| $\mathbb{I}_A(.)$ | indicator function of the set $A$ |
| $\partial_x = \frac{\partial}{\partial x}$ | partial derivative with respect to $x$ |
| $\Lambda$ | lattice (discrete index set) |
| $E \subseteq \mathbb{N}_0^\Lambda$ | state space of the process |
| $\eta_x \in \mathbb{N}_0$ | number of particles at a given site $x$ |
| $\eta = (\eta_x : x \in \Lambda)$ | configuration of the process |
| $\eta^{x \to y}$ | configuration after a particle jumped from $x$ to $y$ |
| $N \in \mathbb{N}_0$ | Total number of particles in the system |
| $L \in \mathbb{N}$ | Fixed number of lattice sites $L = \|\Lambda\|$ if $\Lambda$ is finite |
| $f_n \simeq g_n$ | Asymptotically equal i.e. $f_n = g_n + o(1)$ |
| $D[0, \infty)$ | set of right continuous paths with left limits (path space) (p.5) |
| $(\eta(t) : t \geq 0)$ | the process on $D[0, \infty)$ (p.10) |
| $(S(t) : t \geq 0)$ | Markov semigroup (p.7) |
| $\mathcal{L}$ | Markov generator of the process (p.8) |
| $\mathbb{P}^\eta$ | probability measure on path space with initial condition $\eta(0) = \eta$ (p.5) |
| $\mathbb{E}^\eta$ | expectation value with respect to $\mathbb{P}^\eta$ (p.5) |
| $\pi, \pi_{L,N}$ | stationary distribution on $E$ (p.13) |
| $\chi$ | mobility (p.19) |

| | |
|---|---|
| $\nu_\phi$ | grand canonical measures (p.17) |
| $z(\phi)$ | partition function, normalisation of $\nu_\phi$ (p.19) |
| $\phi \geq 0$ | fugacity parameter (p.18) |
| $\phi_c$ | radius of convergence of $\phi \mapsto z(\phi)$ (p.18) |
| $\rho$ | particle density in the thermodynamic limit (p.20) |
| $\rho_c$ | critical density (p.20) |
| $\sigma^2(t)$ | second moment $\sigma^2(t) = \mathbb{E}[\eta_x^2(t)]$ (p.64) |
| $m_i(t)$ | $i^{\text{th}}$ moment, $i \in \mathbb{N}_0$ (p.43) |
| $c(k,l)$ | jump rates of the misanthrope-type process (p.17) |
| $g(k)$ | jump rates of the zero-range process (p.17) |
| $q(x,y)$ | jump probabilities from site $x$ to $y$ (p.17) |
| $d$ | diffusion parameter for the inclusion process (p.17) |
| $D(\eta,\xi)$ | duality function or self-duality function (p.15) |

# Chapter 1

# Introduction

Nowadays, there is a significant increase in approaches to global challenges facing science, business and society using mathematics. Real world problems, for example, “When is there a jam on motorways?”, “Will the infection die out?”, “Can everyone in the world have blue eyes?” , “Do the rich get richer?”, can be studied using complex stochastic modelling. The main aim is to understand and predict emergent behaviour on macroscopic scales in terms of the microscopic dynamics and interactions of individual components. The probabilistic approximation of the system is sufficient, since a precise description of all microscopic details is impractical due to the large number of objects involved. Here, the macroscopic measurable quantities of the system correspond to expected values of certain observables, such as density or flux of the system.

In this thesis, we consider stochastic particle systems consisting of randomly moving particles on a lattice, whose motion is influenced by interactions between each other [90, 72]. Mathematically, these can be regarded as a continuous time Markov process on a discrete state space where the rules of interactions can be adapted to represent microscopic phenomena of interest. The concept of particles is to be understood in the most general sense, simply representing discrete degrees of freedom such as cars on motorway, people infected by a certain disease, a genotype in genes, or a price of asset orders in financial markets. Qualitative changes in the behaviour of these systems under variation of system parameters are known as collective phenomena or phase transitions, and condensation transition is our particular interest.

To describe a large system, taking the mean-field limit possibly turns a difficult model into a much simpler one. Basically, we replace the structure of the lattice by a very large complete graph geometry, interactions between individuals are approximated by averaged interactions. In particular, phase transitions can often already be observed in the mean-field model so studying this limit is a wise first step in the analysis of an even more complicated model. The mean-field kinetic equations originate from the idea of the Boltzmann equation which describes the evolution of dilute gases in a large system of particles [58]. The mean-field scaling can also be thought of as particles rarely colliding and only interacting after sufficient time to de-correlate. A famous example is the Vlasov equation, describing the electrons interacting through Coulomb force which was introduced in [94]. Mean-field equations are also widely used for particle exchange models [8] and exchange-driven growth models [12]. In the mean-field type equations, gelation can appear as a singularity in the solution, corresponding to the divergence of a cluster of arbitrarily large size in finite time (for examples see [71, 1, 77]).

Population dynamics is the study of how and why populations change in size and structure over time. A fruitful and diverse tool to study population dynamics is the theory of birth death processes. This theory was well-developed from the beginning of the twentieth century as a result of attempts to model growth of a population with stochastic demographic factors. The first and simplest birth death processes considered by Yule [97] and Kendall [65] provide a natural and useful theoretical framework for modern biology. The theory has been fairly well-developed so that analysis and application are used widely in both mathematics and biology [83], and in our case can be used to analyse the dynamics of mean-field equations for stochastic particle systems.

Stochastic particle systems studied in this thesis are of misanthrope type [26], where at most one particle is allowed to jump at a time and the rate that this occurs depends on the number of particles in the arrival and departure sites. We focus on the system defined on lattices with local conservation of the number of particles. Under certain conditions on the rates, the models exhibit stationary product measures which are amenable to be analysed and have furnished our understanding of condensation transitions [24] where a finite fraction of all particles in the system concentrates on a vanishing volume fraction, often only a single site. First, we study the zero-range process, which is a model with no restriction on the occupation numbers and the jump rate only depends on the number of particles on the departure site (zero-range interaction). It serves as a generic model for condensation and jamming transitions in simple traffic models (recent results and applications can be found in [33, 34, 49] and references therein). Another model is the inclusion process with an additional attractive mechanism, where now the jump rates depend on both departure and arrival sites. It was originally introduced as a dual process of a heat conduction model [41], and then further developed as a bosonic counterpart of the exclusion process in [43]. Besides its application in energy transportation, this model can also be interpreted as a multi-allele version of the Moran model [81] which is a well-known model in population genetics for finite populations.

Recently, research interest has focused on understanding the dynamic properties of condensing stochastic particle systems. Before reaching stationarity with the condensate concentrating on a single lattice site, the dynamics of the condensed phase exhibits a coarsening process with a decreasing number of cluster sites of increasing size. The coarsening behaviour in condensing systems has been studied heuristically first in [29] and in various other models [47, 45, 39], including in spatially heterogeneous ones [49]. There is a significant literature on the dynamics of condensation of the zero-range process [44, 55, 49, 46] and for the inclusion process [18] on a heuristic level, and very few rigorous results [53, 10].

In this thesis, we aim to investigate such dynamics in the general cases, both heuristically and rigorously in the mean-field geometry, where implementations of simulations have been inspired by population dynamics models. We develop a rigorous derivation of mean-field equations for stochastic particle systems. The single site dynamics of stochastic particle systems in the limit of diverging system size converges to a Markovian birth death chain. These results provide a contribution towards a rigorous understanding of coarsening behaviour in the thermodynamic limit of diverging system size and particle number. We generalise recent methods of explicit scaling solutions for the condensed part of the single site distribution for zero-range processes and inclusion process.

The thesis is organised as follows: In Chapter 2 we give the general setting of the models, mathematical tools used in this thesis and a brief overview of results related to our work. In Chapter 3, we prove the convergence of the limiting single site dynamics of a misanthrope-type process with bounded rates on a complete graph to a Markovian non-linear birth death chain, described by a mean-field equation. In Chapter 4, we investigate the coarsening dynamics in condensing zero-range processes and provide the size-biased birth death chains as a new tool to study the dynamics. In Chapter 5, we extend our results we found in Chapter 3 and 4 to inclusion processes and derive some exact results through duality. The last chapter is devoted to a summary of the thesis.

# Chapter 2

# Basic theory of stochastic particle systems

In this chapter, we give precise mathematical definitions of the stochastic particle systems that are presented in this thesis and summarise the previous results which are relevant to our models.

In Section 2.1, we introduce the basic general setting, standard notations in general stochastic particle systems, stating some key definitions which are used frequently in this thesis, such as generator, master equation, and stationary measure, largely following [72] and [73]. In Section 2.2, we review results on a family of interacting particle systems of misanthrope type, focusing on zero-range processes and inclusion processes which we will study in detail later.

## 2.1 Definitions and properties

### 2.1.1 Markov processes

One might be familiar with the idea of Markov Processes where given the present state, the future time evolution does not depend on its past. In this thesis, the *state space* of the process denoted by $E$ is either a compact or locally compact separable metric space. We consider both cases and write $C(E)$ to be either the space of continuous real-valued functions in the compact case or the space of continuous real-valued functions vanishing at infinity[1]in the locally compact case. To make $C(E)$ a

[1]A function $h$ on a locally compact space is said to be *vanishing at infinity* if for every $\epsilon > 0$, there is a compact set $K \in E$ such that $|h(\eta)| < \epsilon$ for all $\eta \notin K$.

Banach space, we use the uniform norm:

$$\|f\| = \sup_{\eta \in E} |f(\eta)|.$$

We denote the *continuous time stochastic process* by $(\eta(t) : t \geq 0)$ on the countable state space $E = S^{\Lambda}$ where $S \subseteq \mathbb{N}_0$ is a countable set and $\Lambda$ is a finite lattice. We have $\eta = (\eta_x : x \in \Lambda)$ where $\eta_x \in S$ is interpreted as the number of particles at site $x$ and $S$ may be restricted to finite subsets. In this case, we can equip $E$ with the discrete topology which is simply the power set and obtain a compact metric space which then makes $E$ a compact metric space as well. The measurable structure on $E$ is given by the $\sigma$-algebra of Borel sets $\mathcal{B}(E)$ where we simply take discrete topology on $S$ and product topology on $E$ in general case.

The time evolution of the process is given by a sample path $\eta : [0, \infty) \to E$, where the path space $D[0, \infty)$ is the set of the right continuous functions $\eta : [0, \infty) \to E$ with left limits. The $\sigma$-algebra $\mathcal{F}$ on $E$ is the smallest such that the mapping $\eta(\cdot) \mapsto \eta(s)$ is measurable for each $s \geq 0$. For $t \in [0, \infty)$, let $\mathcal{F}_t$ be the smallest $\sigma$-algebra so that all functions $\eta(\cdot) \mapsto \eta(s)$ for $0 \leq s \leq t$ are measurable. The filtered space $(D[0, \infty), \mathcal{F}, \mathcal{F}_t)$ is normally the generic choice for the probability space of the process.

**Definition 2.1.** A *Markov process* with state space $E$ is defined as a family of probability measures $(\mathbb{P}^{\eta} : \eta \in E)$ on $D[0, \infty)$ with a right continuous filtration $(\mathcal{F}_t, t \geq 0)$ [2] on $(D[0, \infty), \mathcal{F})$ with respect to which the process $\eta(t)$ is adapted, satisfying:

(a) $\mathbb{P}^{\eta}[\eta(0) = \eta] = 1$, for all $\eta \in E$.

(b) The mapping $\eta \mapsto \mathbb{P}^{\eta}(A)$ is measurable for every $A \in \mathcal{F}$.

(c) Markov property :

$$\mathbb{E}^{\eta}[G \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}^{\eta(s)} G - \ \mathbb{P}^{\eta} \text{ a.s.} \tag{2.1}$$

for every $\eta \in E$ and all bounded measurable $G$ on $D[0, \infty)$. Here $(\theta_s \omega)(t) = \omega(t + s)$ for all $\omega \in D[0, \infty)$ is the time shift operator.

The Markov property (c) implies that the process is (time-) homogeneous but it can be generalised to be inhomogeneous (see e.g. [52]). With (b), we can start the

[2] A a right continuous filtration $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$ for all $t$.

process with an arbitrary initial distribution $\mu$ where

$$\mathbb{P}^{\mu} = \int \mathbb{P}^{\eta}\mu(d\eta), \text{ and coresponding expectation } \mathbb{E}^{\eta}.$$

Since we focus on countable state spaces $E$, we can define the *transition function* which is a function defined for $t \geq 0$ and $\eta, \eta' \in E$ as

$$p_t(\eta, \eta') = \mathbb{P}^{\eta}[\eta(t) = \eta']. \tag{2.2}$$

It satisfies

$$p_t(\eta, \eta') \geq 0, \quad \sum_{\eta' \in E} p_t(\eta, \eta') = 1, \quad \lim_{t \to 0^+} p_t(\eta, \eta) = p_0(\eta, \eta) = 1, \tag{2.3}$$

and the Chapman-Kolmogorov equations

$$p_{s+t}(\eta, \eta') = \sum_{\xi \in E} p_t(\eta, \xi) p_x(\xi, \eta'). \tag{2.4}$$

The dynamics are characterised by *transition rates* $c(\eta, \eta') \geq 0$ for all $\eta, \eta' \in E$ describing the rate that the system changes from the present state $\eta$ to the future state $\eta'$ in a small time window $dt$ i.e.

$$\mathbb{P}^{\eta}[\eta(dt) = \eta'] = c(\eta, \eta')dt + o(dt) \text{ as } dt \to 0^+ \text{ for } \eta' \neq \eta. \tag{2.5}$$

With (2.2), we have

$$c(\eta, \eta') = \frac{d}{dt} p_t(\eta, \eta')\big|_{t=0} \text{ for } \eta' \neq \eta.$$

As usual for Markov processes, we can write transition rates in a $Q$-matrix as $Q = (c(\eta, \eta') : \eta, \eta' \in E)$ with $c(\eta, \eta) = -\sum_{\eta' \neq \eta} c(\eta, \eta')$. A $Q$-matrix is a collection $c(\eta, \eta')$ of real numbers indexed by $\eta, \eta' \in E$ that satisfies

$$c(\eta, \eta') \geq 0 \text{ for } \eta \neq \eta' \text{ and } \sum_{\eta' \in E} c(\eta, \eta') = 0.$$

All processes we consider in this thesis will have $|c(\eta, \eta)| < \infty$, i.e. do not have instantaneous states (see Section 2.4 in [73] for details).

### 2.1.2 Semigroup and generator

Let us now introduce Feller processes, a particular subclass of Markov processes which includes several notable families such as the standard Brownian motion, Poisson processes, and Lévy processes. We can write our transition function in terms of a family of operators such that

$$S(t)h(\eta) := \mathbb{E}^{\eta}[h(\eta(t))] = \sum_{\eta' \in E} p_t(\eta, \eta')h(\eta'), \quad \text{for} \quad h \in C(E) \tag{2.6}$$

where $\mathbb{E}^{\eta}$ denotes the expectation given that the process starts at $\eta$.

**Definition 2.2.** A Markov process $(\eta(t) : t \geq 0)$ on a state space $E$ is a *Feller process* if $h \in C(E)$ implies $S(t)h \in C(E)$ for all $t \geq 0$, where $S(t) : C(E) \to C(E)$ is the operator (2.6).

**Definition 2.3.** A family $S = (S(t) : t \geq 0)$ of linear operators on $C(E)$ is called a *Markov Semigroup* if it satisfies the following conditions:

(a) Identity at 0 : $S(0)h = h$, for all $h \in C(E)$.

(b) Strong continuity : $t \mapsto S(t)h$ is right-continuous for all $h \in C(E)$.

(c) Markov property : $S(t + s) = S(t)S(s)$ for all $s, t \geq 0$.

(d) Conservation of probability :
If $E$ is compact, $S(t)1 = 1$ for all $t \geq 0$.
If $E$ is locally compact, there exist $h_n \in C(E)$ such that $\sup_n \|h_n\| < \infty$ and $S(t)h_n \to 1$ pointwise as $n \to \infty$ for each $t \geq 0$.

(e) Positivity : $S(t)h \geq 0$ for all non-negative $h \in C(E)$.

Note that (c) is the Markov property analogously to the Chapman-Kolmogorov equations, and is called the semigroup property. Consequently, the semigroup is commutative. Using (d) together with (e), we have

$$\|S(t)h\| \leq \|h\| \cdot \|S(t)1\| \leq \|h\| \text{ for all } h \in C(E),$$

so that each $S(t)$ is a contraction operator. This with (b) and (c) implies that the function $t \mapsto S(t)h$ from $[0, \infty)$ to $C(E)$ is continuous.

**Theorem 2.1.** *Given a Feller process. Then,* $S(t)$ (2.6) *is a semigroup on* $C(E)$ *according to Definition 2.3.*

*Proof.* Omitted. See Theorem 3.15 in [73]. □

Markov semigroups are in one-to-one correspondence with Markov processes $(\mathbb{P}^\eta : \eta \in E)$ as in the following theorem :

**Theorem 2.2.** *Suppose $(S(t) : t \geq 0)$ is a Markov semigroup on $C(E)$ as given in Definition 2.3. Then, there exists a unique Feller process $(\mathbb{P}^\eta : \eta \in E)$ on $E$ such that* (2.6) *holds for all $t \geq 0$.*

*Proof.* Omitted. See Theorem 1.5 in [72]. □

Due to Theorem 2.2, for any given Markov process, the corresponding Markov semigroup can fully describe the time evolution of the expected values of observable $S(t)h \in C(E)$. Hence, rather than explicitly giving the measures $(\mathbb{P}^\eta : \eta \in E)$ for each process, we introduce an operator which gives enough information to determine the process and this will mainly be used in this thesis. This operator is called Markov generator of the Markov semigroup which is denoted by $\mathcal{L}$ and can be defined as the following :

**Definition 2.4.** A *Markov generator* is a linear operator $\mathcal{L}$ on $C(E)$ satisfying:

(a) The domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is dense in $C(E)$.

(b) If $h \in \mathcal{D}(\mathcal{L})$, for any $\alpha \geq 0$, and set $g = h - \alpha\mathcal{L}h$, then

$$\inf_{\eta\in E} h(\eta) \geq \inf_{\eta\in E} g(\eta).$$

(c) The range of the operator $\mathcal{R}(I - \alpha\mathcal{L}) = C(E)$ for all sufficiently small $\alpha > 0$, where $I : C(E) \to C(E)$ is the identity map.

(d) If $E$ is compact, $1 \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}1 = 0$.
If $E$ is locally compact, for small $\lambda > 0$, there exist $f_n \in \mathcal{D}(\mathcal{L})$ so that $g_n = f_n - \lambda\mathcal{L}f_n$ satisfies

$$\sup_n \|g_n\| < \infty \text{ and both } f_n, g_n \to 1 \text{ pointwise as } n \to \infty.$$

The following well-known *Hille-Yosida* theorem provides a one-to-one correspondence between Markov semigroups and Markov generators.

**Theorem 2.3.** *There is a one-to-one correspondence between Markov generators on $C(E)$ and Markov semigroups on $C(E)$. This correspondence is given by :*

*(a)* $D(\mathcal{L}) = \left\{ h \in C(E) : \textit{the strong limit} : \lim_{t \to 0^+} \frac{S(t)h - h}{t} \textit{ exists} \right\}$, *and*

$$\mathcal{L}h = \lim_{t \to 0^+} \frac{S(t)h - h}{t} \quad \textit{for } h \in D(\mathcal{L}). \tag{2.7}$$

*(b)* $S(t)h = \lim_{n \to \infty} (I - \frac{t}{n}\mathcal{L})^{-n} h$ *for* $h \in C(E)$ *and* $t \geq 0$.

*Proof.* Omitted. See Theorem 2.9 in [72]. □

**Corollary 2.1.** *Let* $S(t)$ *be a semigroup generated by* $\mathcal{L}$*. Then, the following hold: For each* $h \in D(\mathcal{L})$,

$$S(t)h \in D(\mathcal{L}),$$

*and*

$$\mathcal{L}S(t)h = S(t)\mathcal{L}h \quad \textit{for each } t \geq 0. \tag{2.8}$$

*In addition, for each* $h \in D(\mathcal{L})$,

$$\frac{d}{dt} S(t)h = \mathcal{L}S(t)h = S(t)\mathcal{L}h, \tag{2.9}$$

*which is called the backward and forward equations respectively.*

*Proof.* Let $h \in D(\mathcal{L})$ and fix $t \geq 0$. Then, for $s \geq 0$,

$$\begin{aligned} \mathcal{L}S(t)h &= \lim_{s \to 0^+} \frac{(S(s)S(t)h - S(t)h)}{s} = \lim_{s \to 0^+} \frac{(S(t)S(s)h - S(t)h)}{s} \\ &= \lim_{s \to 0^+} S(t) \frac{S(s)h - h}{s}. \end{aligned}$$

As $s \to 0^+$, the right-hand side converges to $S(t)(\mathcal{L}h)$ since $h \in D(\mathcal{L})$ and $S(t)$ is continuous on $E$. This gives $S(t)h \in D(\mathcal{L})$ and $\mathcal{L}S(t)h = S(t)\mathcal{L}h$ as required.
For $h > 0$, we have

$$\begin{aligned} \lim_{s \to 0^+} \frac{S(t+s)h - S(t)h}{h} &= \lim_{s \to 0^+} \frac{S(t)S(s)h - S(t)h}{s} \\ &= \lim_{s \to 0^+} \left( \frac{S(s) - 1}{s} \right) S(s)h \\ &= \mathcal{L}S(t)h = S(t)\mathcal{L}h, \end{aligned}$$

since $S(t)h \in D(\mathcal{L})$. □

Here, we can write $S(t) = e^{t\mathcal{L}}$ as for every $h \in C(E)$, $S(t)$ is the solution of

(2.8). In this thesis, the forward equation is commonly used in computations via

$$\frac{d}{dt}\mathbb{E}^{\eta}[h(\eta)(t)] = \mathbb{E}^{\eta}[(\mathcal{L}h)(\eta(t))]. \tag{2.10}$$

For a finite system, Markov processes and semigroups are characterised by the $Q$-matrix as defined before. To make the connection with the generator :

**Proposition 2.1.** *Consider a Markov process $(\eta(t) : t \geq 0)$ with a countable state space $E$ with transition rates $c(\eta, \eta')$. The generator is given by*

$$\mathcal{L}h(\eta) = (Qh)(\eta) = \sum_{\eta' \in E} c(\eta, \eta')(h(\eta') - h(\eta)). \tag{2.11}$$

*Proof.* From (2.6) and (2.7), we have

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{S(t)h(\eta) - h(\eta)}{t} \\
&= \lim_{t\to 0^+} \frac{\mathbb{E}^{\eta} h(\eta(t)) - h(\eta)}{t} \\
&= \lim_{t\to 0^+} \frac{\sum_{\eta' \in E} \mathbb{P}^{\eta}(\eta(t) = \eta')h(\eta') - h(\eta)}{t} \\
&= \lim_{t\to 0^+} \frac{\left(\sum_{\eta \neq \eta'} c(\eta, \eta')h(\eta')t + h(\eta)\left(1 - \sum_{\eta \neq \eta'} c(\eta, \eta')t\right) + o(t)\right) - h(\eta)}{t} \\
&= \sum_{\eta' \in E} c(\eta, \eta')(h(\eta') - h(\eta)).
\end{aligned}$$

□

**Example 2.1.** Consider a birth death process (BD) $(X_t : t \geq 0)$ which is a continuous time Markov process with state space $E = \mathbb{N}_0$, where there are at most two possible state transitions: "birth" and "death" which increases and decreases (respectively) the state by one. The value $X_t \in \mathbb{N}_0$ can be thought of as the population size at time $t$. Its $Q$-matrix is given by

$$c(k, l) = \begin{cases} \beta_k & \text{if} \quad l = k + 1, \\ \mu_k & \text{if} \quad l = k - 1, \\ -(\beta_k + \mu_k) & \text{if} \quad l = k, \\ 0 & \text{otherwise}. \end{cases} \tag{2.12}$$

The birth rate and the death rate satisfy $\beta_k \geq 0, \mu_k \geq 0$ and $\mu_0 = 0$. By Proposition

2.1, the generator is given by

$$\mathcal{L}_{\mathrm{BD}}h(k) = \mu_k(h(k-1) - h(k)) + \beta_k(h(k+1) - h(k)), \tag{2.13}$$

for all $h \in C(E)$.

Note that time until the next birth is an exponential random variable $T_B \sim \exp(\beta_k)$ and, independently, the time until the next death is then $T_D \sim \exp(\mu_k)$. The rates for holding times $T = \min\{T_B, T_D\}$ are given by $\beta_k + \mu_k$. The embedded chain for the birth death process is just a simple random walk with state dependent up probability $\frac{\beta_k}{\beta_k+\mu_k}$ and down probability $\frac{\mu_k}{\beta_k+\mu_k}$.

### Master equation

There is another equivalent approach to describe Markov processes with discrete state space $E$ which is widely used in the physics literature, namely the *master equation*. It can be derived from the generator and the semigroup of Markov processes. Denote the distribution on $E$ at time $t$ starting from the initial distribution $\mu$ as

$$p_t[\eta] = \mathbb{P}^{\mu}[\eta(t) = \eta] = \int_E S(t)\mathbb{I}_\eta d\mu,$$

where $\mathbb{I}_\eta$ is the indicator function on $E$. Indeed, $p_t[\eta] = \sum_{\eta' \in E} p_t(\eta', \eta)p_0[\eta']$ with transition function given in (2.2). The time evolution of expected values of an observable can be derived by substituting it into the forward equation in (2.10) to get

$$\begin{aligned}\frac{d}{dt}p_t[\eta] &= \int_E S(t)\mathcal{L}\mathbb{I}_\eta d\mu \\ &= \sum_{\xi\in E} p_t[\xi] \sum_{\xi'\in E} c(\xi, \xi')(\mathbb{I}_\eta(\xi') - \mathbb{I}_\eta(\xi)).\end{aligned}$$

Hence, we have the master equation :

$$\frac{d}{dt}p_t[\eta] = \sum_{\xi\in E}(p_t[\xi]c(\xi, \eta) - p_t[\eta]c(\eta, \xi)). \tag{2.14}$$

The terms on the right-hand side can be interpreted as gain and loss dynamics, respectively.

### Martingales

Markov processes can also be characterised through martingales, which we will explain in the following.

**Definition 2.5.** An $\mathbb{R}^n$-valued process $(M(t) : t \geq 0)$ is said to be a *martingale* with respect to a filtration $(\mathcal{F}_t : t \geq 0)$ if

$$\mathbb{E}[|M_t|] < \infty \quad \text{and} \quad \mathbb{E}[M(t)|\mathcal{F}_s] = M(s)$$

for all $t \geq 0$ and $s < t$.

**Theorem 2.4.** *Consider a Feller process $(\eta(t) : t \geq 0)$ with semigroup $S(t)$ and generator $\mathcal{L}$. Then, for every $h \in \mathcal{D}(\mathcal{L})$,*

$$M(t) = h(\eta(t)) - h(\eta(0)) - \int_0^t \mathcal{L}h(\eta(s))ds \tag{2.15}$$

*is a martingale with respect to the natural filtration of the process for every $\eta(0) \in E$.*

*Proof.* Omitted. See Theorem 3.32 in [73]. □

**Theorem 2.5.** *Suppose the Feller process $(\eta(t) : t \geq 0)$ has semigroup $S(t)$ and generator $\mathcal{L}$. If $\mathbb{P}$ is a probability measure on $D[0, \infty)$ such that*

*(a) $\mathbb{P}[\eta(0) = \eta] = 1$, and*

*(b) for every $h \in \mathcal{D}(\mathcal{L})$, the process* (2.15) *is a martingale with respect to $\mathbb{P}$.*

*Then, $\mathbb{P} = \mathbb{P}^\eta$, i.e. $\mathbb{P}$ is equal to the path measure of the process as given in Definition 2.1.*

*Proof.* Omitted. See Theorem 3.33 in [73]. □

In other words, the martingale property determines the measure $\mathbb{P}^\eta$. Martingales are processes with constant expectation and no drift. The size of fluctuations of a Martingale is characterised by the quadratic variation denoted by $[M](t)$ in the sense that $M^2(t) - [M](t)$ is a martingale and thus $\mathbb{E}[M^2(t)] - \mathbb{E}[M^2(0)] = \mathbb{E}[[M](t)]$ for all $t \geq 0$. For the martingale (2.15), it can be calculated as

$$[M](t) = \int_0^t [\mathcal{L}h^2(\eta(s)) - 2h(\eta(s))\mathcal{L}h(\eta(s))]ds.$$

**Example 2.2.** A Poisson process $(N(t) : t \geq 0)$ with rate $\lambda$ is defined on $E = \mathbb{N}_0$ by the generator

$$\mathcal{L}h(n) = \lambda(h(n+1) - h(n)).$$

Then, with $h(n) = n$ and $h(n) = n^2$ we have

$$\mathcal{L}n = \lambda \text{ and } \mathcal{L}n^2 = \lambda(2n+1),$$

respectively. Hence, we get

$$M(t) = N(t) - \lambda t \text{ is a martingale with quadratic variation } [M](t) = \lambda t.$$

### Stationary measures

A stationary distribution for a process is, in short, a probability distribution which is invariant under the dynamics of the process.

**Definition 2.6.** A measure $\nu$ on $E$ is said to be *stationary* if

$$\nu(S(t)h) = \nu(h) \text{ for all } t \geq 0 \text{ and } h \in C(E),$$

where we use the notation $\nu(h) = \int_E h d\nu$.

**Proposition 2.2.** *A measure $\nu$ on $E$ is stationary if and only if*

$$\nu(\mathcal{L}h) = 0 \text{ for all } h \in D(\mathcal{L}).$$

*Proof.* Omitted. See Proposition 2.3 in [72]. □

Not every Markov process has stationary distributions or if it has, it is not necessarily unique.

**Definition 2.7.** A Markov process $(\eta(t) : t \geq 0)$ with semigroup $(S(t),\ t \geq 0)$ is *ergodic* if there exists a unique stationary distribution $\pi$, and

$$p_t(\eta, \cdot) \to \pi \text{ for all initial conditions } \eta \in E.$$

**Definition 2.8.** A Markov process $(\eta(t) : t \geq 0)$ is *irreducible* if for all $\eta, \eta' \in E$,

$$p_t(\eta, \eta') > 0 \text{ for some } t > 0.$$

Irreducibility of a Markov process means that all states in the state space can be reached from any state. If the state space is compact, i.e. in particular if it is finite, then an irreducible Markov process is ergodic.

**Definition 2.9.** A measure $\nu$ is *reversible* with respect to the semigroup $(S(t) : t \geq 0)$ if

$$\nu(fS(t)g) = \nu(gS(t)f) \ \text{ for all } f, g \in C(E),$$

or equivalently in terms of generator,

$$\nu(f\mathcal{L}g) = \nu(g\mathcal{L}f) \ \text{ for all } f, g \in C(E).$$

It is easy to see that every reversible measure is stationary, by simply choosing $g = 1$ in the case of compact $E$. For a Markov process with transition rate $c$ on a countable state space $E$, a measure $\nu$ is reversible if and only if it satisfies the *detailed balance* conditions

$$\nu(\eta)c(\eta, \xi) = \nu(\xi)c(\xi, \eta) \ \text{ for all } \eta, \xi \in E. \tag{2.16}$$

**Example 2.3.** Recall the birth death process given in Example 2.1. If $\beta_0 = 0$, then 0 is the absorbing state and the process is not irreducible. For birth death process with strictly positive rate $\beta_k$ and $\mu_k$ ($\mu_0 = 0$), we can find stationary distribution $\pi$ using detailed balance as in (2.16) i.e. $\pi_k\beta_k = \pi_{k+1}\mu_{k+1}$ for all $k \in \mathbb{N}_0$. Then

$$\pi_k = \frac{\beta_0\beta_1 \cdots \beta_{k-1}}{\mu_1\mu_2 \cdots \mu_k}\pi_0, \ k \geq 1, \tag{2.17}$$

and if

$$\sum_{k=0}^{\infty} \frac{\beta_0\beta_1 \cdots \beta_{k-1}}{\mu_1\mu_2 \cdots \mu_k} < \infty, \tag{2.18}$$

$\pi$ can be normalized by

$$\pi_0 = \left( \sum_{k=0}^{\infty} \frac{\beta_0\beta_1 \cdots \beta_{k-1}}{\mu_1\mu_2 \cdots \mu_k} \right)^{-1}. \tag{2.19}$$

### 2.1.3 Duality

In the late 1940s, duality of Markov processes with respect to a duality function has been introduced in [76, 63, 70] and this concept has been further developed in a series of papers [89, 57, 25, 93, 32]. Duality is a tool to compute properties of one process in terms of another. It has been applied in many fields such as mathematical population genetics, where the backward in time dual process is given by a coalescence (see e.g.

[79, 30, 36, 13] for detail), and also for interacting particle systems where it has been recently developed in [42, 19]. In spite of this wide interest, there is no complete theory for the duality of Markov processes with respect to a function, even some basic questions, such as conditions for existence of a dual process for a given Markov process, have not yet been fully resolved. The relation between stationary measures and duality functions was recently found in [86] which helps to construct self-duality functions for such processes. For the general theories in this section, we refer to [59] where the existence and uniqueness of dual processes through functional analysis have been studied in detail.

**Definition 2.10.** Suppose $\eta = (\eta(t) : t \geq 0)$ and $\xi = (\xi(t) : t \geq 0)$ are two Markov processes with state spaces $E_1$ and $E_2$, respectively. The process $\eta$ is said to be *dual* to process $\xi$ with respect to the duality function $D \in C(E_1 \times E_2)$, if

$$\mathbb{E}^\eta D(\eta(t), \xi) = \mathbb{E}^\xi D(\eta, \xi(t)), \tag{2.20}$$

for all $\eta \in E_1, \xi \in E_2$ and $t \geq 0$.

We can use the duality relationship (2.20) in computing the expectations of certain condition functions for a process of interest $(\eta(t) : t \geq 0)$ in terms of the expectations of a second auxiliary process $(\xi(t), t \geq 0)$, which is often considerably simpler than the process of interest. We can relate the definition of duality functions with the Markov semigroups and generator as well. Let $(S^\eta(t) : t \geq 0)$ and $(S^\xi(t) : t \geq 0)$ be the Markov semigroups of the process $(\eta(t) : t \geq 0)$ and $(\xi(t) : t \geq 0)$, respectively. We can write (2.20) as

$$(S^\eta(t)D(\cdot, \xi))(\eta) = (S^\xi(t)D(\eta, \cdot))(\xi), \tag{2.21}$$

for all $\eta \in E_1$ and $\xi \in E_2$.
Duality implies that in fact for every $t$, and $0 \leq s \leq t$,

$$\begin{aligned}(S^\eta(t)D(\cdot, \xi))(\eta) &= (S^\eta(s)S^\eta(t-s)D(\cdot, \xi))(\eta) \\ &= (S^\eta(s)S^\xi(t-s)D(\eta, \cdot))(\xi),\end{aligned}$$

using the Chapman-Kolmogorov equation and duality property, respectively. Note that (2.21) is given by the special case $s = 0$.
The equivalent version for the generators is formulated in the next result.

**Proposition 2.3.** *Let $(\eta(t) : t \geq 0)$ and $(\xi : t \geq 0)$ be Markov processes on $E_1$ and $E_2$, with generators $\mathcal{L}^\eta$ and $\mathcal{L}^\xi$ with domains $\mathcal{D}(\mathcal{L}^\eta)$ and $\mathcal{D}(\mathcal{L}^\xi)$, respectively.*

*Suppose $D : E_1 \times E_2 \to \mathbb{R}$ is in $C(E_1 \times E_2)$. If $D(\eta, \cdot)$, $S^\xi(t)D(\eta, \cdot) \in \mathcal{D}(\mathcal{L}^\xi)$ for all $\eta \in E_1$, $t \geq 0$ and $D(\cdot, \xi)$, $S^\eta(t)D(\cdot, \xi) \in \mathcal{D}(\mathcal{L}^\eta)$ for all $\xi \in E_2$, $t \geq 0$, then*

$$\mathcal{L}^\eta D(\cdot, \xi)(\eta) = \mathcal{L}^\xi D(\eta, \cdot)(\xi) \qquad \text{for all } \eta \in E_1,\ \xi \in E_2 \tag{2.22}$$

*if and only if $\eta$ and $\xi$ are dual with respect to $D$.*

*Proof.* Omitted. See Proposition 1.2 in [59]. □

#### 2.1.3.1 Self-Duality

A Markov process is called self-dual with respect to a duality function $D$ if the equation (2.20) holds for any two copies of the process $(\eta(t) : t \geq 0)$ and $(\xi(t) : t \geq 0)$. These two processes are defined on the same state space and governed by the same dynamics, but with different initial conditions, where in one copy there may be infinitely many particles but in the other with only a small number (e.g. 1 or 2) of particles.

**Definition 2.11.** Let $\eta = (\eta(t) : t \geq 0)$ and $\xi = (\xi(t) : t \geq 0)$ be two copies of the same Markov processes on a state space $E$. The process is said to be *self-dual* with self-duality function $D : E \times E \to \mathbb{R}$ if for all $\eta \in E$ and $\xi \in E$, (2.20) holds.

Note that with Proposition 2.3 on finite state space $E$ this is equivalent to

$$QD = DQ^T, \tag{2.23}$$

where $Q^T$ is the transpose of the $Q$-matrix.

## 2.2 Interacting Particle System

In 1970, a variety of interacting particle systems has been introduced by Spitzer in [90]. Throughout this thesis, we only consider processes with conservation of particles (i.e. no particle being created or annihilated) and study models defined on finite lattices in the limit of large system size instead of the infinite lattices directly. Therefore, we have compact or locally compact state space $E$, and can define the particle system using the general theory in Section 2.1.2. On infinite lattices the space $E$ is not necessarily locally compact and the definition of the dynamics is challenging for unbounded local state space; there is no general theory to guarantee a well-defined process yet, it requires further regularity assumptions on the rates [2, 7]. As long as the local state space $S$ is finite, $E = S^\Lambda$ is compact for all $\Lambda$

and the Feller construction applies [73]. This can be used for exclusion processes or other classical interacting particle systems as discussed in [72]. The misanthrope-type process we introduce next is defined on finite lattices with locally compact $E$, so the standard method applied.

### 2.2.1 Definition and Properties

Interacting particle systems are continuous time Markov processes on a discrete state space $E = S^\Lambda$, where $\Lambda$ is the *lattice* which is a countable set such as the regular lattice or the vertex set of a graph and $S \subseteq \mathbb{N}_0$ is the local state space. We focus on finite lattices, so $E$ is locally compact or compact if $S$ is finite. *Configurations* are typically denoted by $\eta = (\eta_x : x \in \Lambda)$, where $\eta_x \in \mathbb{N}_0$ is the number of particles on site $x$. A *stochastic particle system of misanthrope type* [26] with $S = \mathbb{N}_0$ is then denoted by $(\eta(t) : t \geq 0)$ and its dynamics is given by the infinitesimal generator

$$(\mathcal{L}h)(\eta) = \sum_{x,y \in \Lambda} q(x,y) c(\eta_x, \eta_y)(h(\eta^{x \to y}) - h(\eta)), \tag{2.24}$$

where the usual notation $\eta^{x \to y}$ indicates a configuration where one particle has moved from site $x$ to $y$ i.e. $\eta_z^{x \to y} = \eta_z - \delta(z,x) + \delta(z,y)$ and $\delta$ is the Kronecker delta. The functions $h \in C(E)$ are continuous functions as introduced earlier in Section 2.1.1 and $h$ can be regarded as an observable. The $q(x,y) \geq 0$ are irreducible, finite range transition probabilities of a single random walker on $\Lambda$ with $q(x,x) = 0$. To ensure that the process is non-degenerate, the jump rates satisfy

$$\begin{cases} c(0,l) = 0 & \text{for all } l \geq 0 \\ c(k,l) > 0 & \text{for all } k > 0 \text{ and } l \geq 0. \end{cases} \tag{2.25}$$

There are several processes with this dynamics [24], for example:

1. zero-range processes (ZRP) : $c(k,l) = g(k)$ with $g : \mathbb{N}_0 \to [0, \infty)$ such that $g(k) = 0$ iff $k = 0$,
2. inclusion processes (IP) : $c(k,l) = k(d + l)$ with parameter $d \geq 0$,
3. explosive condensation processes (ECP) : $c(k,l) = k^\lambda(d + l^\lambda)$ with parameters $\lambda > 1$ and $d \geq 0$.

A misanthrope-type process with generator (2.24) has a family of stationary product measures (see e.g. [26, 35, 85]) denoted by

$$\nu_\phi = \prod_{x \in \Lambda} \nu_\phi[\eta_x = \cdot], \tag{2.26}$$

provided that

$$\frac{c(k,l)}{c(l+1,k-1)} = \frac{c(k,0)c(1,l)}{c(l+1,0)c(1,k-1)} \quad \text{for all } k \geq 1, l \geq 0, \tag{2.27}$$

and if one of the two following conditions holds

$$q(x,y) = q(y,x) \ \text{ for all } x,y \in \Lambda, \quad \text{or} \tag{2.28}$$

$$c(k,l) - c(l,k) = c(k,0) - c(l,0) \ \text{ for all } k,l \geq 0. \tag{2.29}$$

The marginals are given explicitly by

$$\nu_\phi[\eta_x = n] = \frac{1}{z(\phi)} w(n)\phi^n \quad \text{with} \quad w(n) = \prod_{k=1}^{n} \frac{c(1,k-1)}{c(k,0)}, \tag{2.30}$$

and normalization given by the partition function

$$z(\phi) := \sum_{n=0}^{\infty} w(n)\phi^n \ . \tag{2.31}$$

The parameter $\phi \geq 0$ is the fugacity controlling the average particle density

$$R(\phi) := \sum_{n=0}^{\infty} n\nu_\phi[\eta_x = n], \tag{2.32}$$

which is a monotone increasing function of $\phi$ with $R(0) = 0$. It can also be calculated via

$$R(\phi) = \phi\partial_\phi \log z(\phi),$$

as $z(\phi)$ in (2.31) is a generating function.

The stationary distributions exist if $z(\phi)$ is finite. We define the domain of (2.26) by

$$\mathcal{D} = \{\phi \geq 0 : z(\phi) < \infty\}.$$

The domain is of the form $\mathcal{D} = [0,\phi_c]$ or $[0,\phi_c)$ (the right boundary of the domain depends on the particular process), where

$$\phi_c = (\limsup_{n\to\infty} w(n)^{1/n})^{-1}$$

is the radius of convergence of $z(\phi)$.

Another important quantity to understand the large-scale dynamics is the mobility

$$\chi(\phi) = \mathbb{E}_{\nu_\phi}[c(\eta_x, \eta_y)] \text{ for } x \neq y, \tag{2.33}$$

i.e. the expected jump rate under the stationary distribution. Note that this is independent of $x \neq y \in \Lambda$ since the stationary measures are permutation invariant in homogeneous system. In an asymmetric system, this determines the stationary current together with $q(x, y)$, and is related to the diffusivity in symmetric systems (see e.g. [14]).

**Condensation**

Under certain conditions on the jump rates, stochastic particle systems can exhibit a condensation transition where a non-zero fraction of all particles accumulates in a condensate, provided the particle density exceeds a critical value $\rho_c$ (see Definition 2.12 below). The system phase separates into a condensate and a homogeneous background. The homogeneous background is distributed according to the maximal invariant measure with critical density and the excess mass concentrates on a subextensive part of the lattice, establishing the condensed phase. In finite systems with stationary product measures, the condensed phase occupies only a single lattice site, which is located uniformly at random on the lattice. Condensing models with homogeneous stationary product measures have attracted significant research interest (see e.g. [24, 35] for recent summaries), including zero-range processes of the type introduced in [29, 33], inclusion processes with a rescaled system parameter [54, 18] and explosive condensation models [95, 21]. The role of spatial inhomogeneities and their interplay with particle interactions is summarised in detail for zero-range processes in [49] (see also [24] and [78] for further references), and in this thesis we only focus on spatially homogeneous processes. While the stationary measures have been understood in great detail on a rigorous level [62, 55, 6, 4, 24], the dynamics of these processes continues to pose interesting mathematical questions. First recent results for zero-range and inclusion processes have been obtained on metastability in the stationary dynamics of the condensate location [11, 5, 15], approach to stationarity on fixed lattices under diverging particle density [54, 10], a hydrodynamic limit for density profiles below the critical value [91], and also in the context of related population models [28].

Formally, we consider the misanthrope-type process (2.24) restricted on the finite lattice of size $|\Lambda| = L$ and with conserved particle number. With periodic boundary conditions, it is translation invariant and irreducible on the state space

$$E_{L,N} = \{\eta \in E : \sum_{x\in\Lambda} \eta_x = N\}, \tag{2.34}$$

for any fixed number of particles $N \in \mathbb{N}_0$. Since the total number of particles is conserved and is the only conserved quantity under the dynamics, the process is ergodic with a unique stationary measure supported on $E_{L,N}$ which is given explicitly by

$$\pi_{L,N} = \nu_\phi[\cdot \mid \sum_{x\in\Lambda} \eta_x = N].$$

This is independent of the fugacity $\phi$ so we can choose $\phi = 1$ for simplicity. This together with (2.30) leads to

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \mathbb{I}(\eta \in E_{L,N}), \tag{2.35}$$

with $Z_{L,N} = \sum_{\eta\in E_{L,N}} \prod_{x\in\Lambda} w(\eta_x)$ the normalization.

Here, $\{\pi_{L,N} : N \in \mathbb{N}_0\}$ are called the canonical measures, whereas $\{\nu_\phi : \phi \in \mathcal{D}\}$ are the grand-canonical measures (see e.g. [24]). As for the product measure $\nu_\phi$ where we define the mobility (2.33), we define the canonical mobility as

$$\chi_{L,N} := \mathbb{E}_{\pi_{L,N}}[c(\eta_x, \eta_y)].$$

In the thermodynamic limit

$$L, N \to \infty \text{ such that } \frac{N}{L} \to \rho \geq 0,$$

the product measure (2.26) are usually a good approximation to the sequence of $\pi_{L,N}$, which is called the equivalence of ensembles in statistical mechanics. Since we know that the average particle density $R(\phi)$ (2.32) is strictly increasing with $R(0) = 0$, we can define the critical limit as follows.

**Definition 2.12.** The critical density $\rho_c \in [0, \infty]$ is defined as

$$\rho_c := \lim_{\phi\to\phi_c} R(\phi) \in [0, \infty]. \tag{2.36}$$

Therefore, $\rho_c$ is the maximal density for the family of product measures to occur. If $\mathcal{D} = [0, \phi_c)$ then $\rho_c = \infty$ so there is no condensation (see [68]). And if $\rho_c < \infty$ the model exhibits a condensation transition (see e.g. [24]).

### 2.2.2 Graphical representation of interacting particle systems

It is natural and useful to construct interacting particle systems via a probabilistic or graphical representation which was introduced by Harris in 1972 [56]. It is constructed explicitly in terms of collections of independent Poisson processes. The graphical construction played a crucial role in many proofs for various stochastic particle systems [74]. There are many advantages to this approach such as the possibility of constructing the process starting from different initial configurations on the same probability space. It also explains duality in an explicit way as can be observed in the space-time graphical picture, the evolution of the dual process is then seen by reversing the time direction [59]. We only consider the construction for bounded rates $c(k,l) \leq \bar{C}$ in Chapter 3, where it takes a particularly simple form. A Poisson process of rate $\bar{C}q(x,y)$ is associated with each pair of sites $x, y \in \Lambda$. At the event times of the Poisson process, a particle jumps from site $x$ to site $y$ with probability $c(\eta_x, \eta_y)/\bar{C} \leq 1$. This again constructs all processes with arbitrary initial configurations on the probability space of the Poisson processes. With this construction, the system is realised as a collection of interacting copies of simple Poisson processes.

**Example 2.4** (Contact process)**.** The contact process is a model of an interacting particle systems usually interpreted as the spread of an infection. The process is defined on the state space $E = \{0,1\}^\Lambda$ of particle configurations $(\eta_x : x \in \Lambda)$, which is restricted to maximally only one particle per site. Here, $\eta_x = 1$ means site $x$ is occupied (interpreted as infected) and $\eta_x = 0$ means site $x$ is empty (interpreted as healthy). Then, the dynamics of the process is characterised by the generator

$$\mathcal{L}_{\text{contact}} h(\eta) = \sum_{x \in \Lambda} \Big( \eta_x + (1-\eta_x) \sum_{y \in \Lambda} q(y,x) \eta_y \Big) (h(\eta^x) - h(\eta)), \tag{2.37}$$

where $\eta^x$ is the configuration where the state of site $x$ is flipped i.e. $(\eta^x)_y = \eta_y(1 - \delta_{x,y}) + (1-\eta_x)\delta_{x,y}$. Here, infected sites recover or become healthy at a constant rate 1, while a healthy site gets infected independently with rate $q(y,x) > 0$ by infected sites $y \in \Lambda$. This process can be constructed using the graphical representation as shown in Figure 2.1 for $q(x,y) = \lambda(\delta_{y,x+1} + \delta_{y,x-1})$ with infection rate $\lambda > 0$ in one dimension. We will use this idea of the graphical construction later on in Section 3.3.2.

### 2.2.3 Zero-range process

The zero-range process (ZRP) was first introduced by Spitzer in [90]. It is a stochastic particle system with no restriction on the number of particles per site, and a

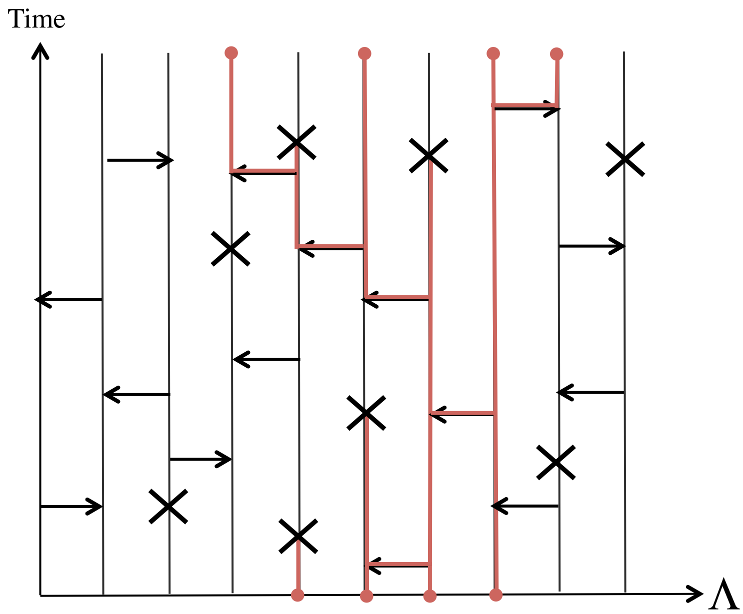



Figure 2.1: Graphical construction of the contact process (2.37). "×" $\sim PP(1)$ are given by iid Poisson processes with rate 1 on each line marking the recovery events. The infection events are marked by "$\rightarrow$" and "$\leftarrow$" which indicate the independent Poisson processes $PP(\lambda)$ for infection events between nearest neighbours. Examples of infected sites are indicated in red.

particle jumps to another site with a jump rate depending only on the occupation of the departure site. The model has attracted research interest as it exhibits simple product stationary measures [90, 2] and a condensation transition in some particular cases [33, 44, 55]. In one dimension, the zero-range process can be mapped to an exclusion process (another model in which lattice sites can either be occupied by a single particle or vacant), where particles in ZRP can be thought as vacancies in the exclusion process, and sites in the ZRP as occupied sites in the exclusion process [34, 75]. In this section, we give a definition and summarise the key properties of the zero-range process, for further details see [90, 68].

### Definition

First, define the *jump rates* $g : \mathbb{N}_0 \to [0, \infty)$ as a non-negative function of the number of particles $\eta_x$ at site $x$. Then, the dynamics of the zero-range process $(\eta(t) : t \geq 0)$ on $E$ is defined by the infinitesimal generator acting on a test function $h \in C(E)$, substituting $c(k, l) = g(k)$ in (2.24):

$$(\mathcal{L}_{\text{ZRP}} h)(\eta) = \sum_{x,y \in \Lambda} g(\eta_x) q(x, y) (h(\eta^{x \to y}) - h(\eta)). \tag{2.38}$$

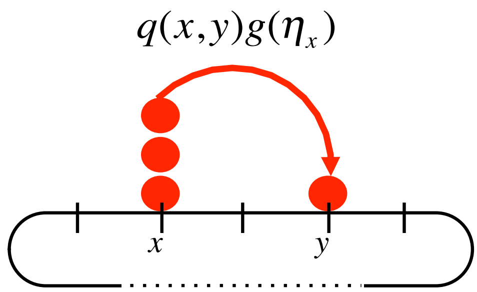



Figure 2.2: Dynamics of zero-range process. Particles perform random walks with rate $q(x,y)g(\eta_x)$, which is independent of number of particles on target site $y$.

The jump rates $g(\eta_x)$ are assumed to be strictly positive on positive integers to ensure that the process is non-degenerate and irreducible, and

$$g(k) = 0 \text{ iff } k = 0.$$

### Stationary measures

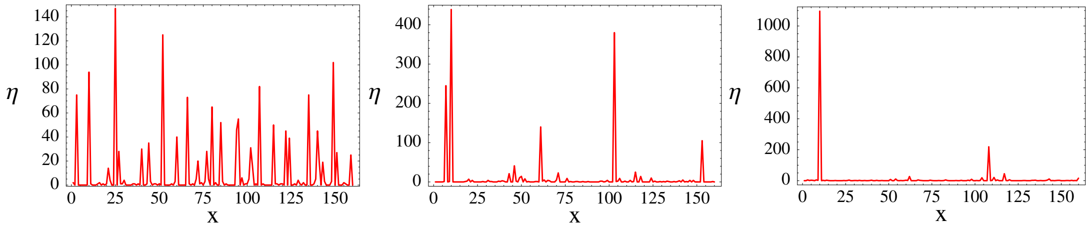
On finite lattices $\Lambda$, the state space $E$ is locally compact and the construction of the dynamics given in Section 2.1 applies. Note that ZRPs fulfil (2.27) and (2.29), so they exhibit product stationary measures with the marginal as given in (2.30), where the stationary weight is now given by

$$w(n) = \prod_{k=1}^{n} \frac{1}{g(k)}. \tag{2.39}$$

The activity for zero-range processes is simply given by

$$\chi(\phi) := \mathbb{E}_{\nu_\phi}[g(\eta_x)] = \frac{1}{z(\phi)} \sum_{n=0}^{\infty} g(n)w(n)\phi^n = \phi, \tag{2.40}$$

which can be calculated directly using the form of the stationary weights $w(n)$ given in (2.39). To construct the process on infinite $\Lambda$ further regularity assumptions on $g$ are necessary (see e.g. [2, 7]).




(a) Coarsening (b) Saturation (c) Stationarity

Figure 2.3: Illustration of the condensation dynamics via sample density profiles for the symmetric zero-range process with homogeneous initial conditions. In the coarsening regime (a) clusters exchange particles and large ones grow at the expense of small ones. Clusters do not change location, and with only few cluster sites left the system saturates (b). This leads to the stationary state with a single condensate (c) which changes position only on a slower time scale. Note the different ranges of the $y$-axes.

### Condensation in zero-range processes

In this thesis, we focus on zero-range processes introduced in [29, 33] with monotonic decreasing jump rates of the form

$$g(k) = \begin{cases} 0 & \text{if} \quad k = 0, \\ 1 + \frac{b}{k^\gamma} & \text{otherwise,} \end{cases} \tag{2.41}$$

for any constant $b > 0$ and $\gamma \in (0, 1]$.
It is known that this system exhibits condensation if $\gamma \in (0, 1)$ or $\gamma = 1$ and $b > 2$ [33, 55]. The stationary weights for $n \to \infty$ are

$$w(n) \sim \begin{cases} n^{-b} & \text{if} \quad \gamma = 1 \text{ and } b > 2, \\ \exp\left(-\frac{bk^{1-\gamma}}{1-\gamma}\right) & \text{if} \quad \gamma \in (0,1) \text{ and } b > 0. \end{cases} \tag{2.42}$$

The stationary distributions exist for all $\phi \in [0, 1]$ with the maximal invariant measure $\nu_1$ and critical density $\rho_c = R(1) < \infty$. When the imposed particle density $\rho$ in a finite system of size $L$ exceeds $\rho_c$, the system phase separates into a homogeneous background distributed according to $\nu_1$ with critical density, and the condensate where the excess mass $(\rho - \rho_c)L$ concentrates on a single randomly located lattice site. The condensing zero-range process with rates (2.41) exhibits particularly interesting dynamics starting from homogeneous initial conditions with a particle density $\rho > \rho_c$, which have first been described in [44, 48] for $\gamma = 1$. The stages of the dynamics can be summarised as follows (cf. also Figure 2.3):

1. Nucleation regime: the density decreases locally to $\rho_c$ and the resulting excess

mass concentrates into cluster sites. Outside clusters on so-called bulk sites, the system relaxes to its stationary distribution $\nu_{\phi_c}$.

2. Coarsening regime: the clusters exchange particles through the bulk, leading to a decreasing number of cluster sites of increasing size.

3. Saturation Regime: on finite systems, eventually there is only a single cluster site left which contains all excess particles and forms the condensate. This is reached on a slower timescale which is $L^{1+\gamma}$ for a complete graph or asymmetric system of size $L$ depending on the parameter $\gamma$. In one-dimensional symmetric systems, the scale is $L^{2+\gamma}$.

4. The remaining single condensate changes position on an even longer time scale, which is $L^{1+b}$ for $\gamma = 1$ [48, 11, 5], and expected to be of stretched exponential form for $\gamma \in (0, 1)$.

Our main interest in this thesis is the coarsening regime and we will explain how to effectively describe the dynamics in Chapter 4.

### 2.2.4 Inclusion process

In this section we introduce another interesting model called the inclusion process (IP) which was originally found in [41] as a dual of a heat conduction model. It is a system of particles performing independent random walks on a lattice with an additional attractive (inclusion) interaction between each other. It has later been developed as bosonic counter part of the exclusion process [43]. The most important property in our context is that, like the zero-range process, the inclusion process can exhibit a condensation transition in the limit of vanishing diffusion parameter. The inclusion interaction then plays an important role and leads to typical stationary configurations where a single lattice site contains almost all the particles in the system, which has been established rigorously on finite lattices [53, 54].

#### Definition

The dynamics of inclusion process $(\eta(t) : t \geq 0)$ with parameter $d > 0$ on $E$ is defined by the infinitesimal generator acting on a bounded test function $h$, where $c(k, l) = k(d + l)$ with a parameter $d \geq 0$ in (2.24)

$$(\mathcal{L}_{\text{IP}} h)(\eta) = \sum_{x,y \in \Lambda} q(x, y) \eta_x (d + \eta_y)(h(\eta^{x \to y}) - h(\eta)). \tag{2.43}$$

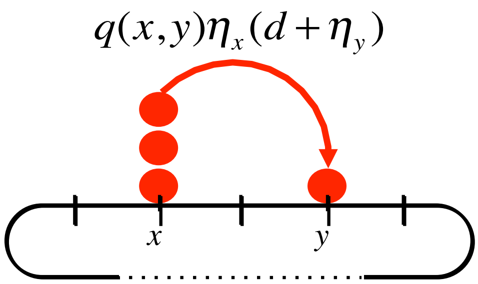



Figure 2.4: Dynamics of inclusion process. Particles perform random walks with rate $q(x,y)d\eta_x$ and attract each other with rate $q(x,y)\eta_x\eta_y$.

The parameter $d$ determines the relative rate of the independent random walk of particles in comparison to the interacting inclusion part given by the product $\eta_x\eta_y$.

#### Stationary measure

The stationary measures of the inclusion process were identified in [41]. The rates fulfil conditions (2.27) and (2.29). Translation invariant systems may exhibit stationary measures of product form with the marginal as given in (2.30) and stationary weights

$$w(n) = \frac{\Gamma(d+n)}{n!\Gamma(d)} \sim n^{d-1} \text{ as } n \to \infty, \tag{2.44}$$

where $\Gamma$ denotes the Gamma function. The single site partition function is of the form

$$z(\phi) = \sum_{n=0}^{\infty} w(n)\phi^n = (1-\phi)^{-d}. \tag{2.45}$$

This implies that $z(\phi)$ diverges as $\phi \to 1$, hence the measures only exist for all $\phi \in [0,1)$. The average particle density is then

$$R(\phi) = \phi\partial_\phi \log z(\phi) = \frac{d\phi}{1-\phi}.$$

For inclusion processes, the mobility can be simply written as a function of the density $\rho = R(\phi)$ as

$$\chi(\phi) = R(\phi)(d + R(\phi)).$$

#### Condensation in inclusion processes

On the complete graph, the simplest case for the inclusion process is when $d = 0$, which obviously leads to absorbing states. For diffusion parameter $d$ which scales with the system size, i.e. $d = d_L$, the condensation in the inclusion process occurs when $d_L$ decays fast enough such that $d_L \ll 1/L$, and in the condensed regime all particles will concentrate on a single site [22, 18]. For diffusion parameter which scales depending on $N$ i.e. $d = d_N \to 0$ as $Nd_N \to \infty$, the diffusion gets slow and the attractive inclusion interaction creates condensates. The dynamics of the rescaled process in this case was studied in details in [53].

In this thesis, we focus on the system with diffusion parameter $d \to 0$ in thermodynamic limit. We will see in Chapter 5 that the analysis of the case $d = 0$ gives a good approximation on complete graphs. The critical density in this case is $\rho_c = 0$. The condensate contains all particles which can be localised on any site of the lattice, so it is an absorbing state for the dynamics.

### Self-dual property of the inclusion process

From Definition 2.11 of self-duality, the symmetric inclusion process is self-dual with the duality function

$$D(\xi, \eta) = \prod_x d(\xi_x, \eta_x) \quad \text{where } d(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma(d)}{\Gamma(d+k)}. \tag{2.46}$$

The self-duality is then given by

$$\mathbb{E}^\eta[D(\xi, \eta(t))] = \mathbb{E}^\xi[D(\xi(t), \eta], \tag{2.47}$$

which was proved in [42, 43].

### Connection to population genetics

In population genetics, the basic source of randomness due to reproduction is considered. This leads to so-called random genetic "drift", which is considered as a major evolutionary force in addition to selection [36]. The forward evolution of population genetics models has a long history started in the 1920s; Wright and Fisher are considered as two of the founders in this area [96, 40]. Here, we give the basic example of population genetics models and their connection. The classical Wright-Fisher model is a discrete-time model of a population with constant size $N$ and only two types i.e. $E = \{1, 2\}^N$. The Moran model is a continuous-time version of the Wright-Fisher model [80, 81]. The two-allele Moran model $(Z_t : t \geq 0)$ where $Z_t$ denotes the number of alleles type 1 at time $t$ on state space $E = \{0, ..., N\}$ can be written in the generator form as

$$(\mathcal{L}^N_{\text{Moran}} h)(i) = i(N - i)(h(i + 1) + h(i - 1) - 2h(i)), \tag{2.48}$$

for all $i \in E$ with $h(-1) = h(N+1) = 0$.
The interpretation is that each of the $N$ individuals reproduces with rate 1, and then one of the $N+1$ individual is killed instantaneously. The diffusion limit of this model can be obtained by using generator and space rescaling in a very large population size. We rescale the state space of the process to be a Markov process on $\{0, \frac{1}{N}, ..., 1\} \subseteq [0,1]$ i.e. $z_t^N := \frac{1}{N} Z_t$. The generator (2.48) becomes for a function $h : [0,1] \to \mathbb{R}$

$$(\mathcal{L}^N_{\text{Moran}} h)\left(\frac{i}{N}\right) = i(N-i)\left[\frac{1}{N^2} h''\left(\frac{i}{N}\right) + O(1/N^3)\right],$$

and for $z \in [0,1]$,

$$(\mathcal{L}^N_{\text{Moran}} h)(z) \to z(1-z)h''(z) \text{ as } N \to \infty.$$

Hence, as $N \to \infty$, the generator $\mathcal{L}^N_{\text{Moran}}$ of the Moran model converges to the well-known Wright-Fisher diffusion [30] :

$$(\mathcal{L}_{\text{WFdiff}} h)(z) = \frac{1}{2} z(1-z)h''(z), \tag{2.49}$$

but run twice faster.
The duality concept has been used to connect these two processes. The backward problems are obtained from looking at the forward process backward in time [50]. The corresponding backward processes have studied in the 1980s, with Kingman [66, 67] being the best-known. By applying duality function $H(z,j) = z^j$ to (2.49) we get

$$\begin{aligned}
\mathcal{L}_{\text{WFdiff}} H(\cdot, j)(z) &= \frac{1}{2} z(1-z)[j(j-1)z^{j-2}] \\
&= \frac{j(j-1)}{2}[H(z, j-1) - H(z,j)] \\
&= \mathcal{L}^j_{\text{Kingman}} H(z, \cdot)(j),
\end{aligned}$$

where the process with generator

$$\mathcal{L}^j_{\text{Kingman}} h(j) = \frac{j(j-1)}{2}(h(j-1) - h(j)),$$

is known as Kingman's coalescent. It is a pure death process on the state space $\mathbb{N}$ with absorbing state 1.

The inclusion process can be interpreted as a multi-allele version of the Moran model with additional mutation, which describes generic dynamics in finite populations where sites are interpreted as allele types and particles as individuals or people. For detail on how to include mutation see e.g. [30, 36]. Condensation corresponds to fixation of a particular species in the limit of vanishing mutation rate, which is effectively given by the parameter $d$ [54]. On a fixed lattice $\Lambda$ (representing the allele type) in the limit of large population size $N \to \infty$, the dynamics is giving by a multi-dimensional version of the Wright-Fisher diffusion [54]. It has also been shown in [20] that there is a limiting duality between Wright-Fisher diffusion and the Moran model which corresponds to the inclusion process as mentioned before. This is duality between two "forward processes" and as we know that inclusion process is also self-dual, this could apply for Moran model as well. The "backward process" also could arise from the limiting multi-type Wright-Fisher diffusion process with mutation which leads to the dual process of lineage coalescence starting from infinity similar to Kingman's coalescence but with additional lost due to mutation rate [92, 51, 31].

# Chapter 3

# Derivation of mean-field equations for stochastic particle systems

## 3.1 Introduction

In this chapter, we study stochastic particle systems in a limit of large system size leading to a description by a mean-field master equation for the time evolution of a single lattice site, which has often been used in physics literature [44, 48, 35]. For conservative systems, these equations are very similar to mean-field rate equations in the study of cluster growth models (see e.g. [69] and the references therein). We focus on interacting particle systems where only one particle jumps at a time, which corresponds to monomer exchange in cluster growth models as studied in [12], and also in the well-known Becker-Döring model [9, 8].

While these mean-field equations often provide the starting point for the analysis and have an intuitive form, to our knowledge their connection to underlying particle systems has not been rigorously established so far. Hence, we provide a rigorous derivation of this equation for misanthrope-type processes as defined in (2.24) with bounded jump rates and homogeneous initial conditions on a complete graph. The limit equation describes the dynamics of the fraction $f_k(t) \in [0, 1]$ of lattice sites with a given occupation number $k$, and also provides the master equation of a birth death chain for the limiting single site dynamics of the process. Note that no time rescaling is required and the limiting dynamics are non-linear, i.e. the birth and death rates of the chain depend on the distribution $f_k(t)$. Even though the limiting birth death dynamics is irreducible under non-degenerate initial conditions, the non-linearity leads to conservation of the first moment of the initial distribution, resulting in a continuous family of stationary distributions, as has been observed before for other non-linear birth death chains (see e.g. [64]). To establish the mean-field property in the limit, we show the asymptotic decay of correlations by bounding percolation clusters in the graphical construction of the process with branching processes up to finite times, similar to [3, 38]. Existence of limits follows from standard tightness arguments, and our proof also includes a simple uniqueness argument for solutions of the limit equation. While uniqueness has been established for more complicated coagulation fragmentation models [8], we could not find a result covering our case in the literature.

Our result provides a contribution towards a rigorous understanding of the approach to stationarity in the thermodynamic limit of diverging system size and particle number. This exhibits an interesting coarsening regime characterised by a power-law time evolution of typical observables, which has been identified in previous heuristic results [44, 55, 21, 46] also on finite dimensional regular lattices. Condensation implies that stationary measures for the limiting birth death dynamics only exist up to a first moment $\rho_c$, above which $f_k(t)$ phase separates over time into two parts describing the mass distribution in the condensate and the background of the underlying particle system. Explicit travelling wave scaling solutions for the condensed part of the distribution have been found in [44, 60, 46] for zero-range processes and in [12] for a specific inclusion process, and will be discussed in detail.

This chapter is organised as follows. In Section 3.2 we introduce notations and state our main result with the proof given in Section 3.3. In Section 3.4 we discuss basic properties of the limit dynamics and its solutions, as well as limitations and possible extensions of our result. We also provide a concluding discussion in Section 3.5.

## 3.2 Notation and Main result

Consider a stochastic particle system $(\eta(t) : t > 0)$ of misanthrope type as introduced in Section 2.2.1 on finite lattices $\Lambda$ of size $|\Lambda| = L$ with the infinitesimal generator (2.24) i.e.

$$(\mathcal{L}h)(\eta) = \sum_{x,y\in\Lambda} q(x,y)c(\eta_x,\eta_y)(h(\eta^{x\to y}) - h(\eta)). \tag{2.24 revisited}$$

We focus on complete graph dynamics, i.e. $q(x,y) = 1/(L-1)$ for all $x \neq y$, and denote by $\mathbb{P}^L$ and $\mathbb{E}^L$ the law and expectation on the path space $D[0,\infty)$ of the process. We will study the processes $t \mapsto F_k(\eta(t))$ defined by the test functions

$$F_k(\eta) := \frac{1}{L} \sum_{x \in \Lambda} \delta_{\eta_x, k} \in [0, 1], \tag{3.1}$$

counting the fraction of lattice sites for each occupation number $k \geq 0$. Expectations are denoted by

$$f_k^L(t) := \mathbb{E}^L\big[F_k(\eta(t))\big] = \frac{1}{L} \sum_{x \in \Lambda} \mathbb{P}^L[\eta_x(t) = k] \in [0, 1], \tag{3.2}$$

and we write $f^L(t) = (f_k^L(t) : k \in \mathbb{N}_0)$. Note that $f^L(t)$ are probability distributions on $\mathbb{N}_0$ for all $t \geq 0$. Following (2.14), the time evolution is then given by

$$\frac{d}{dt} f_k^L(t) = \frac{d}{dt} \mathbb{E}\big[F_k(\eta(t))\big] = \mathbb{E}\big[(\mathcal{L} F_k)(\eta(t))\big]. \tag{3.3}$$

As usual this equation is not closed for finite system sizes $L$, since the right-hand side is not a function of $f^L(t)$. Our aim is to derive a closed equation in the limit $L \to \infty$.

In the following, we consider a sequence (in $L$) of initial conditions $(\eta_x(0) : x \in \Lambda)$ of the process such that

$$\{\eta_x(0) : x \in \Lambda\} \quad \text{are i.i.d. with distribution } f^L(0) \text{ for all } L \tag{3.4}$$

and such that there exists a probability distribution $f(0) = (f_k(0) : k \in \mathbb{N}_0)$ with

$$f_k^L(0) \to f_k(0) \text{ for all } k \geq 0 \text{ as } L \to \infty. \tag{3.5}$$

The second condition excludes cases where the sequence $f^L(0)$ is not tight or does not have a unique limit. The simplest choice with the required properties is, of course, a product measure with marginals $f^L(0) = f(0)$ for all $L$. By symmetry of the dynamics on the complete graph, $(\eta_x(t) : x \in \Lambda)$ is therefore permutation invariant for all $t \geq 0$ and we also have

$$f_k^L(t) = \mathbb{P}^L[\eta_x(t) = k] \quad \text{for all} \quad x \in \Lambda. \tag{3.6}$$

We further assume that the jump rates are uniformly bounded with

$$\bar{C} := \sup_{k,l} c(k, l) < \infty. \tag{3.7}$$

Our main theorem can be formulated as a convergence result for the single site dynamics with state space $\mathbb{N}_0$,

$$(\eta_x(t) : t \geq 0) \quad \text{for fixed } x \in \Lambda \text{ (with } \Lambda \text{ big enough).} \tag{3.8}$$

**Theorem 3.1.** *Consider a process with generator* (2.24) *on the complete graph with uniformly bounded rates (3.7) and initial conditions satisfying (3.4) and (3.5). Then, the single site process* (3.8) *converges weakly on path space to a birth death chain with distribution $f(t) = (f_k(t) : k \in \mathbb{N}_0)$ characterised by the **mean-field master equation***

$$\begin{aligned}\frac{df_k(t)}{dt} &= \sum_{l\geq 0} c(k+1,l) f_l(t) f_{k+1}(t) + \sum_{l\geq 0} c(l,k-1) f_l(t) f_{k-1}(t) \\ &\quad - \Big(\sum_{l\geq 0} c(k,l) f_l(t) + \sum_{l\geq 0} c(l,k) f_l(t)\Big) f_k(t) \quad \textit{for all } k \geq 0,\end{aligned} \tag{3.9}$$

*with initial condition $f(0)$ given by (3.5). Here we use the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$ and recall that $c(0,l) = 0$ for all $l \geq 0$.* (3.9) *has a unique solution $(f(t) : t \geq 0)$, and in particular $f_k^L(t) \to f_k(t)$ as $L \to \infty$ for all $t \geq 0$.*

We see that $\frac{d}{dt} \sum_{k\geq 0} f_k(t) = 0$, and with (3.5) the limit is indeed the master equation of a birth death chain with state space $\mathbb{N}_0$, birth rate $\sum_{l\geq 0} c(l,k) f_l(t)$ and death rate $\sum_{l\geq 0} c(k,l) f_l(t)$. Note that the chain and its master equation are non-linear since the birth and death rates depend on the distribution $f(t)$. Further details are provided in Section 3.5.

**Theorem 3.2.** *Consider the process with generator* (2.24) *and conditions as in Theorem 3.1. We have propagation of chaos, i.e. for all $m \geq 1$, and $t \geq 0$ as $L \to \infty$ $(\eta_1(t), \eta_2(t), ..., \eta_m(t))$ converges to independent random variables with distribution $f(t)$ of Theorem 3.1.*

## 3.3 Proof of the main result

The proof follows a standard approach. We first establish existence of limits via a tightness argument, then characterise all limit points as solutions of (3.9) using a coupling to a branching process based on the graphical construction, and finally show that (3.9) has a unique solution for a given initial condition.

### 3.3.1 Existence

**Proposition 3.1.** *Consider the process with generator* (2.24) *and conditions as in Theorem 3.1. Then, the law of the single site process* $\eta_x(t)$ (3.8) *is tight as* $L \to \infty$. *This implies existence of weak limit points* $(f(t) : t \geq 0)$ *of the sequence* $(f^L(t) : t \geq 0)$ *as defined in* (3.2).

*Proof.* For each $L$ large enough, consider the single site process $\eta_x(t)$ for a fixed $x \in \Lambda$ with law $\mathbb{Q}^L$ on the path space $D([0,\infty), \mathbb{N}_0)$. We will show tightness of the sequence $\mathbb{Q}^L$ as $L \to \infty$, which implies existence of limit points $\mathbb{Q}$. Since $f_k^L(t) = \mathbb{Q}^L[\eta_x(t) = k]$, this also provides existence of limit points $t \mapsto f(t) = \mathbb{Q}[\eta_x(t) = \cdot]$.

Interpreting $\eta_x : \eta \mapsto \eta_x$ as a mapping, $\mathbb{Q}^L = \mathbb{P}^L \circ \eta_x^{-1}$ is given as the image measure of $\mathbb{P}^L$ under $\eta_x$. By a version of Aldous' criterion to establish tightness for $\mathbb{Q}^L$ (cf. Theorem 16.10 in [16]), it suffices to show that

$$\lim_{a\to\infty} \limsup_{L\to\infty} \mathbb{P}^L[\eta_x(t) \geq a] = 0, \tag{3.10}$$

and that for any $\epsilon > 0$

$$\lim_{\delta\to 0^+} \limsup_{L\to\infty} \sup_{s<\delta} \sup_{\zeta\in E} \mathbb{P}_\zeta^L\big[|\eta_x(s) - \zeta_x| > \epsilon\big] = 0. \tag{3.11}$$

Here $\zeta \in E$ denotes the initial condition of the original process and $\mathbb{P}_\zeta^L$ the corresponding path measure, while in (3.10) the initial condition is given by i.i.d random variables with asymptotic density $\rho$ as given in (3.4) and (3.5). Using Markov's inequality, we get by conservation of mass

$$\mathbb{P}^L[\eta_x(t) \geq a] \leq \frac{\mathbb{E}^L[\eta_x(t)]}{a} \leq \frac{\rho(1 + o(1))}{a}.$$

Hence, (3.10) follows easily.
For fixed $\zeta$ and $x$ from above, consider the test function $f(\eta) = |\eta_x - \zeta_x|$ to get

$$\begin{aligned}
\mathcal{L}f(\eta) &= \frac{1}{L-1} \sum_{y\neq x} \Big[c(\eta_y, \eta_x)\big(|\eta_x - \zeta_x + 1| - |\eta_x - \zeta_x|\big) \\
&\qquad\qquad + c(\eta_x, \eta_y)\big(|\eta_x - \zeta_x - 1| - |\eta_x - \zeta_x|\big)\Big] \\
&= \frac{1}{L-1}\Big(\sum_{y\neq x} c(\eta_y, \eta_x) - \sum_{y\neq x} c(\eta_x, \eta_y)\Big)(\mathbb{I}_{\eta_x\geq\zeta_x}(\eta) - \mathbb{I}_{\eta_x<\zeta_x}(\eta)),
\end{aligned} \tag{3.12}$$

with standard notation for indicator functions $\mathbb{I}$. By Itô's formula and with $f(\eta(0)) =$

0, we have for any $s > 0$

$$|\eta_x(s) - \zeta_x| = \int_0^s \frac{1}{L-1}\Big(\sum_{y\neq x} c(\eta_y(u), \eta_x(u)) - \sum_{y\neq x} c(\eta_x(u), \eta_y(u))\Big)$$
$$\big(\mathbb{I}_{\eta_x \geq \zeta_x}(\eta(u)) - \mathbb{I}_{\eta_x < \zeta_x}(\eta(u))\big)du \quad + M(s), \tag{3.13}$$

where $(M(s) : s > 0)$ is a martingale. It has quadratic variation

$$[M](s) = \int_0^s \big[\mathcal{L}f^2 - 2f\mathcal{L}f\big](\eta(u))du,$$

and the integrand is easily computed to be

$$\big[\mathcal{L}f^2 - 2f\mathcal{L}f\big](\eta) = \frac{1}{L-1}\Big(\sum_{y\neq x} c(\eta_y, \eta_x) + \sum_{y\neq x} c(\eta_x, \eta_y)\Big).$$

Since the rates are bounded (3.7), we have for the first term in (3.13)

$$\bigg|\int_0^s \frac{1}{L-1}\Big(\sum_{y\neq x} c(\eta_y(u), \eta_x(u)) - \sum_{y\neq x} c(\eta_x(u), \eta_y(u))\Big)du\bigg| \leq 2\bar{C}s \to 0 \tag{3.14}$$

as $s \to 0$, which holds $\mathbb{P}^L_\zeta$-a.s. uniformly in $\zeta \in E$ and in $L$. The same argument applies to the quadratic variation part, where for $s \to 0$ we get

$$[M](s) = \int_0^s \frac{1}{L-1}\Big(\sum_{y\neq x} c(\eta_y(u), \eta_x(u)) + \sum_{y\neq x} c(\eta_x(u), \eta_y(u))\Big)du$$
$$\leq 2\bar{C}s \to 0. \tag{3.15}$$

Almost sure convergence in (3.14) and (3.15) uniformly in $\zeta \in E$ and in $L$ implies (3.11). □

### 3.3.2 Characterization of limit points

**Proposition 3.2.** *Consider the process with generator* (2.24) *and conditions as in Theorem 3.1. Every limit point* $(f(t) : t \geq 0)$ *of Proposition 3.1 satisfies the mean-field rate equation (3.9).*

We first collect some auxiliary results before giving the proof. Recall the standard graphical construction of interacting particle systems studied in Section 2.2.2. Here, it consists of a family of independent Poisson point processes $PP_{xy}\left(\frac{\bar{C}}{L-1}\right)$ for each pair $x \neq y \in \Lambda$. For a given $\eta$, at the jump time of the point process a particle jumps from $x$ to $y$ with probability $c(\eta_x, \eta_y)/\bar{C}$. This is illustrated in Figure 3.1 for

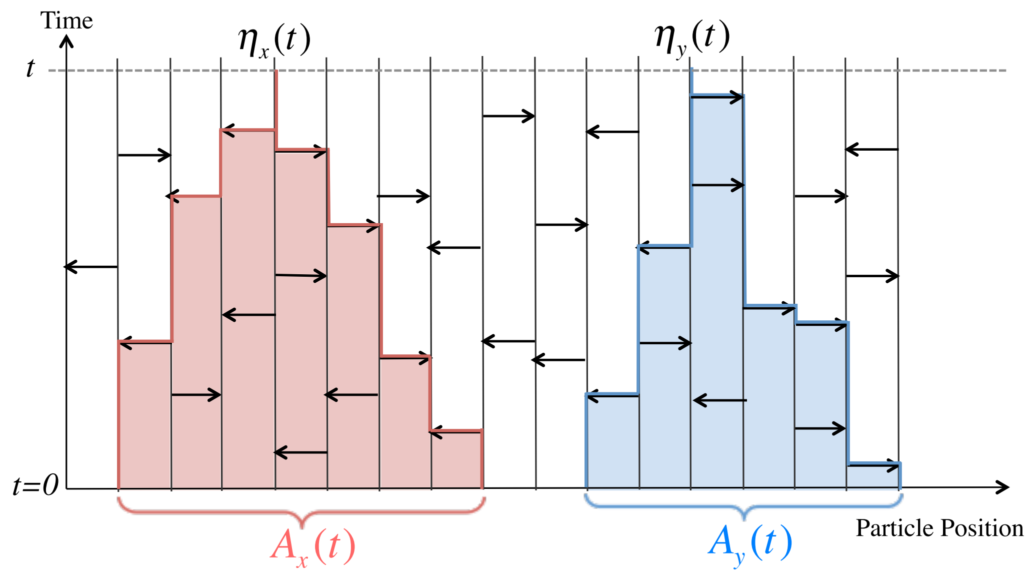



Figure 3.1: Illustration of the graphical construction of the process (2.24) for one-dimensional nearest neighbour dynamics. It is based on independent Poisson processes $PP(\bar{C}/2)$ with jump events shown as $\to$ and $\leftarrow$. The sets $A_x(t)$ and $A_y(t)$ as given in Lemma 3.2, possibly influencing $\eta_x(t)$ and $\eta_y(t)$, respectively, are shown in red and blue.

one-dimensional nearest neighbour dynamics.

We say $(y, 0)$ is connected to $(x, t)$, writing $(y, 0) \to (x, t)$, if there exists a forward in time path along jump events in $\cup_{z,z' \in \Lambda} PP_{zz'}\left(\frac{\bar{C}}{L-1}\right)$ from $(y, 0)$ to $(x, t)$. Equivalently, consider running a contact process on $\Lambda$ without recovery backward in time using all jump events of $\cup_{z,z' \in \Lambda} PP_{zz'}\left(\frac{\bar{C}}{L-1}\right)$ in the time interval $[0, t]$ (regardless of their direction), starting with a single infection at site $x$. Then $(y, 0) \to (x, t)$ if $y$ is infected at time 0. We write

$$A_x(t) = \{y \in \Lambda : (y, 0) \to (x, t)\}, \tag{3.16}$$

for all sites whose configuration at time 0 possibly influences $\eta_x(t)$. We denote the backward in time contact process by

$$\tau \mapsto A_x(t, \tau) := \{y \in \Lambda : (y, t - \tau) \to (x, t)\},\ \tau \in [0, t], \tag{3.17}$$

which is fully determined by the graphical construction of the original process $(\eta_x(t) : t \geq 0)$. We will couple $A_x(t, \tau)$ to auxiliary processes $(\mathcal{N}_x(\tau) : \tau \geq 0)$, contact processes without recovery on $\Lambda$, starting with $\mathcal{N}_x(0) = \{x\}$ and with state dependent infection rates $\frac{\bar{C}}{L-|\mathcal{N}_x(\tau)|}$ across bonds. The total infection rate at any time can be calculated by

$$\mathcal{N}_x(\tau)(L - |\mathcal{N}_x(\tau)|)\frac{\bar{C}}{(L - |\mathcal{N}_x(\tau)|)} = \bar{C}\mathcal{N}_x(t) \ ,$$

and $\mathcal{N}_x = \Lambda$ is absorbing for this process.
Therefore, $(\mathcal{N}_x(\tau) : \tau \geq 0)$ can be coupled to a pure birth process $(N(\tau) : \tau \geq 0)$ with $N(0) = 1$, state space $\mathbb{N}$, and generator

$$\mathcal{L}h(n) = n\bar{C}(h(n+1) - h(n)) \ , \tag{3.18}$$

for $h : \mathbb{N} \to \mathbb{R}$ such that $N(\tau) = |\mathcal{N}_x(\tau)|$ for all $\tau \geq 0$ as long as $N(\tau) < L$.

**Lemma 3.1.** *Let $\mathcal{N}_x(t)$ and $\mathcal{N}_x(t)$ be two independent processes as defined above with $y \neq x$ and denote their law by $\tilde{\mathbb{P}}^L$. Then,*

$$\tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset] \to 1 \ \text{ as } \ L \to \infty, \tag{3.19}$$

*for all fixed $t > 0$.*

*Proof.* Using independence, we have

$$\begin{aligned}\tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) \neq \emptyset] = \sum_{n_x, n_y = 0}^{L} &\tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset \mid |\mathcal{N}_x(t)| = n_x, |\mathcal{N}_y(t)| = n_y] \\ &\cdot \tilde{\mathbb{P}}^L[|\mathcal{N}_x(t)| = n_x]\tilde{\mathbb{P}}^L[|\mathcal{N}_y(t)| = n_y],\end{aligned}$$

and for given sizes $|\mathcal{N}_x(t)| = n_x$ and $|\mathcal{N}_y(t)| = n_y$, $\mathcal{N}_x(t)$, $\mathcal{N}_y(t)$ are independent uniform subsets of $\Lambda$. Therefore,

$$\begin{aligned}&\tilde{\mathbb{P}}^L\left[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset \mid |\mathcal{N}_x(t)| = n_x, |\mathcal{N}_y(t)| = n_y\right] \\ &= \frac{L - n_x - 1}{L - 1} \cdot \frac{L - n_y - 1}{L - 2} \cdots \frac{L - n_x - n_y}{L - n_y} \to 1, \quad \text{as } L \to \infty \ .\end{aligned} \tag{3.20}$$

Consider again the pure birth process $(N(t) : t \geq 0)$ on $\mathbb{N}$ with generator (3.18) with law denoted by $\mathbb{P}$. Then, by construction we have

$$\tilde{\mathbb{P}}^L[|\mathcal{N}_x(t)| = n_x] = \mathbb{I}_{n_x<L}\mathbb{P}[N_x(t) = x] + \mathbb{I}_{n_x=L}\mathbb{P}[N_x(t) \geq L] + (1 - \mathbb{I}_{n_x>L}),$$

for all $t \geq 0$ and the same holds for $\mathcal{N}_y(t)$. Therefore,

$$\tilde{\mathbb{P}}^L[|\mathcal{N}_x(t)| = n_x]\tilde{\mathbb{P}}^L[|\mathcal{N}_y(t)| = n_y] \geq \mathbb{I}_{n_x<L}\mathbb{P}[N_x(t) = x]\mathbb{I}_{n_y<L}\mathbb{P}[N_y(t) = y],$$

and we get

$$
\begin{aligned}
&\tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset] \\
&\geq \sum_{n_x, n_y}^{\infty} \underbrace{\tilde{\mathbb{P}}^L\left[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset \mid |\mathcal{N}_x(t)| = n_x, |\mathcal{N}_y(t)| = n_y\right] \mathbb{I}_{n_x < L} \mathbb{I}_{n_y < L}}_{\to 1 \text{ as } L \to \infty \text{ for all } n_x, n_y \text{ with (3.20)}} \\
&\qquad \cdot \mathbb{P}[N_x(t) = x]\mathbb{P}[N_y(t) = y].
\end{aligned}
$$

Since the $L$-dependent part of the above sum is bounded by 1, we can use dominated convergence to get

$$\tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) = \emptyset] \to 1 \text{ as } L \to \infty,$$

where obviously

$$\sum_{n_x, n_y = 0}^{\infty} \mathbb{P}[N_x(t) = x]\mathbb{P}[N_y(t) = y] = 1,$$

since $N_x(t)$ and $N_y(t)$ are $\mathbb{P} - a.s.$ finite[1]. □

**Lemma 3.2.** *Consider $A_x(t)$ and $A_y(t)$ as defined in* (3.17) *with $y \neq x$. Then, for each fixed $t \geq 0$,*

$$\mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset] \to 1 \quad \textit{as } L \to \infty. \tag{3.21}$$

*Proof.* By the graphical construction, $A_x(t, \tau)$ can be coupled to $\mathcal{N}_x(\tau)$ so that $A_x(t, \tau) \subseteq \mathcal{N}_x(\tau)$ for all $\tau \leq t$ since the infection rates are $\frac{\bar{C}}{L - |\mathcal{N}_x(t)|} \geq \frac{\bar{C}}{L-1}$. The same holds for $(\mathcal{N}_y(\tau) : \tau \geq 0)$ with $A_y(t, \tau) \subseteq \mathcal{N}_y(\tau)$ for all $\tau \leq t$ and in particular $\{A_x(t) \cap A_y(t) \neq \emptyset\} \subseteq \{\mathcal{N}_x(t) \cap \mathcal{N}_y(t) \neq \emptyset\}$. Therefore, with Lemma 3.1 we have

$$\mathbb{P}^L[A_x(t) \cap A_y(t) \neq \emptyset] \leq \tilde{\mathbb{P}}^L[\mathcal{N}_x(t) \cap \mathcal{N}_y(t) \neq \emptyset] \to 0 \text{ as } L \to \infty,$$

and (3.21) follows as required. □

*Proof of Proposition 3.2.* Applying the generator (2.24) with $q(x, y) = 1/(L-1)$ to

[1] A standard computation with generating functions in fact reveals that $N(t)$ has a geometric distribution, where $\mathbb{P}[N(t) = n] = e^{-\bar{C}t}\big(1 - e^{-\bar{C}t}\big)^{n-1}$.

the test function $F_k$, we get

$$
\begin{aligned}
(\mathcal{L}F_k)(\eta) &= \sum_{x,y\in\Lambda} \frac{1}{L-1} c(\eta_x,\eta_y)[F_k(\eta^{x\to y}) - F_k(\eta)] \\
&= -\frac{1}{L-1}\sum_{x,y\in\Lambda} c(k,\eta_y)\frac{\delta_{k,\eta_x}}{L} + \frac{1}{L-1}\sum_{x\in\Lambda} c(k,\eta_x)\frac{\delta_{k,\eta_x}}{L} \\
&\quad - \frac{1}{L-1}\sum_{x,y\in\Lambda} c(\eta_x,k)\frac{\delta_{k,\eta_y}}{L} + \frac{1}{L-1}\sum_{x\in\Lambda} c(\eta_x,k)\frac{\delta_{k,\eta_x}}{L} \\
&\quad + \frac{1}{L-1}\sum_{x,y\in\Lambda} c(\eta_x,k-1)\frac{\delta_{k-1,\eta_y}}{L} - \frac{1}{L-1}\sum_{x\in\Lambda} c(\eta_x,k-1)\frac{\delta_{k-1,\eta_x}}{L} \\
&\quad + \frac{1}{L-1}\sum_{x,y\in\Lambda} c(k+1,\eta_y)\frac{\delta_{k+1,\eta_x}}{L} - \frac{1}{L-1}\sum_{x\in\Lambda} c(k+1,\eta_x)\frac{\delta_{k+1,\eta_x}}{L} \\
&= -\frac{1}{L-1}\sum_{y\in\Lambda} c(k,\eta_y)F_k(\eta) - \frac{1}{L-1}\sum_{x\in\Lambda} c(\eta_x,k)F_k(\eta) \\
&\quad + \frac{1}{L-1}\sum_{x\in\Lambda} c(\eta_x,k-1)F_{k-1}(\eta) + \frac{1}{L-1}\sum_{y\in\Lambda} c(k+1,\eta_y)F_{k+1}(\eta) \\
&\quad + \frac{1}{L-1}\Big( - (B_{L,k} + B'_{L,k}) + B'_{L,k-1} + B_{L,k+1}\Big).
\end{aligned}
\tag{3.22}
$$

Here $B_{L,k} := \sum_x c(\eta_x,k)F_k(\eta)$ and $B'_{L,k} := \sum_x c(k,\eta_x)F_k(\eta)$ are corrections resulting from diagonal terms in the sum over $x,y\in\Lambda$, and are uniformly bounded in $k$ and $L$.

In the following, we will show that $f_k(t)$ fulfills (3.9). From (3.2)-(3.3) and (3.22), we have

$$
\begin{aligned}
\frac{d}{dt} f_k^L(t) &= -\mathbb{E}^L\left[\frac{1}{L}\sum_{y\in\Lambda} c(k,\eta_y)F_k(\eta)\right] - \mathbb{E}^L\left[\frac{1}{L}\sum_{x\in\Lambda} c(\eta_x,k)F_k(\eta)\right] \\
&\quad + \mathbb{E}^L\left[\frac{1}{L}\sum_{x\in\Lambda} c(\eta_x,k-1)F_{k-1}(\eta)\right] + \mathbb{E}^L\left[\frac{1}{L}\sum_{y\in\Lambda} c(k+1,\eta_y)F_{k+1}(\eta)\right] \\
&\quad + O(1/L),
\end{aligned}
\tag{3.23}
$$

where we used that all corrections in (3.22) are of order $1/L$.
To conclude, we will establish that expectations of product terms in (3.23) factorize.

For the second term, we have

$$\begin{aligned}\frac{1}{L}\sum_{x\in\Lambda}c(\eta_x,k)F_k(\eta) &= \sum_{l\geq 0}c(l,k)\frac{1}{L}\sum_{x\in\Lambda}\delta_{\eta_x,l}\frac{1}{L}\sum_{y\in\Lambda}\delta_{\eta_y,k}\\ &= \sum_{l\geq 0}c(l,k)\frac{1}{L^2}\sum_{x,y\in\Lambda}\delta_{\eta_x,l}\delta_{\eta_y,k}\\ &= \sum_{l\geq 0}c(l,k)\frac{1}{L^2}\sum_{x,y\neq x}\delta_{\eta_x,l}\delta_{\eta_y,k}+c(k,k)\frac{1}{L^2}\sum_{x\in\Lambda}\delta_{\eta_x,k}.\end{aligned}$$

Since the rates are bounded (3.7) and $F_k(\eta)\leq 1$, we have

$$\begin{aligned}\mathbb{E}^L\left[\frac{1}{L}\sum_{x\in\Lambda}c(\eta_x,k)F_k(\eta)\right] &= \sum_{l\geq 0}c(l,k)\frac{1}{L^2}\sum_{x\neq y}\mathbb{P}^L[\eta_x(t)=l,\eta_y(t)=k]+O(1/L)\\ &= \sum_{l\geq 0}c(l,k)\mathbb{P}^L[\eta_x(t)=l,\eta_y(t)=k]+O(1/L),\end{aligned}\tag{3.24}$$

where we can fix particular sites $x\neq y$ in the last line by symmetry of the process.

Now, in order to use Lemma 3.2 we write

$$\begin{aligned}\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l\big] &= \mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l,A_x(t)\cap A_y(t)=\emptyset\big]\\ &\quad+\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l,A_x(t)\cap A_y(t)\neq\emptyset\big],\end{aligned}$$

and as $L\to\infty$ we have for the second term

$$\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l,A_x(t)\cap A_y(t)\neq\emptyset\big]\leq\mathbb{P}^L\big[A_x(t)\cap A_y(t)\neq\emptyset\big]\to 0\ .$$

For the first term, we write

$$\begin{aligned}&\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l,A_x(t)\cap A_y(t)=\emptyset\big]\\ &\quad=\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l\mid A_x(t)\cap A_y(t)=\emptyset\big]\mathbb{P}^L\big[A_x(t)\cap A_y(t)=\emptyset\big],\end{aligned}$$

where $\mathbb{P}^L[A_x(t)\cap A_y(t)=\emptyset]\to 1$ as $L\to\infty$ by Lemma 3.2.
Conditional on $\{A_x(t)\cap A_y(t)=\emptyset\}$, the events $\{\eta_x(t)=k\}$ and $\{\eta_y(t)=l\}$ are independent by construction and independence of initial conditions (3.4), and therefore

$$\begin{aligned}&\mathbb{P}^L\big[\eta_x(t)=k,\eta_y(t)=l\mid A_x(t)\cap A_y(t)=\emptyset\big]\\ &\quad=\mathbb{P}^L\big[\eta_x(t)=k\mid A_x(t)\cap A_y(t)=\emptyset\big]\,\mathbb{P}^L\big[\eta_y(t)=l\mid A_x(t)\cap A_y(t)=\emptyset\big]\\ &\quad\to f_k(t)f_l(t)\quad\text{as }L\to\infty.\end{aligned}\tag{3.25}$$

Convergence to the limit points $f_k$ and $f_l$ uses again that the conditional event $\{A_x(t) \cap A_y(t) = \emptyset\}$ has limiting probability 1 with Lemma 3.2. With dominated convergence in (3.24), this implies factorization of

$$\mathbb{E}^L\Big[\sum_{x\in\Lambda} c(\eta_x, k)F_k(\eta)\Big] \to \sum_{l\geq 0} c(l,k) f_l(t) f_k(t), \text{ as } L \to \infty,$$

which follows analogously for the other terms in (3.23). This completes the proof of Proposition 3.2. □

### 3.3.3 Uniqueness

We consider solutions of (3.9) $f(t) = (f_k(t) : k \in \mathbb{N}_0)$ which are limit points of the sequence $f^L(t)$. Since $\sum_{k\geq 0} f_k^L(t) = 1$ and $f_k^L \in [0,1]$ for all $t \geq 0$ and $L$, $f_k(t) \in [0,1]$ for all $t \geq 0$ and $k \geq 0$. Furthermore, Fatou's Lemma implies

$$\|f(t)\|_1 = \sum_{k\geq 0} f_k(t) \leq 1 \quad \text{and therefore} \quad f \in \ell^1.$$

This also implies $f(t) \in \ell^2$ with $\|f(t)\|_2^2 = \sum_{k\geq 0} f_k^2(t) < \infty$.
For given such $f$, the operators

$$(Q_f g)_k = \sum_{l\geq 0} f_k c(k,l) g_l \quad \text{and} \quad (\hat{Q}_f g)_k = \sum_{l\geq 0} f_k c(l,k) g_l$$

are defined for all $g \in \ell^1 \cap \ell^2$. Since $0 \leq c(k,l) \leq \bar{C} < \infty$, we have for non-negative $g, h \geq 0$,

$$\langle g, Q_f h\rangle = \sum_{k,l} g_k f_k c(k,l) h_l \leq \bar{C}\langle g, h\rangle, \tag{3.26}$$

and the same holds for $\hat{Q}_f$.

**Proposition 3.3.** *Let $t \mapsto f(t)$ be a solution to (3.9) with $f_k(t) \in [0,1]$ and $\sum_k f_k(t) \leq 1$ for all $k \geq 0, t \geq 0$. Then, $t \mapsto f(t)$ is unique.*

*Proof.* Suppose $f$ and $\hat{f}$ are two solutions of (3.9) with above the properties and

$f(0) = \hat{f}(0)$. With the convention $f_{-1} = \hat{f}_{-1} \equiv 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\|f-\hat{f}\|_2^2 &= 2\sum_{k\geq 0}(f_k-\hat{f}_k)\frac{d}{dt}\sum_{k\geq 0}(f_k - \hat{f}_k) \\
&= 2\sum_{k\geq 0}(f_k-\hat{f}_k)\Bigg[\sum_{l\geq 0} c(l,k-1)(f_{k-1}f_l-\hat{f}_{k-1}\hat{f}_l) \\
&\qquad + \sum_{l\geq 0} c(k+1,l)(f_{k+1}f_l-\hat{f}_{k+1}\hat{f}_l) \\
&\qquad - \sum_{l\geq 0}(c(k,l)+c(l,k))(f_kf_l-\hat{f}_k\hat{f}_l)\Bigg],
\end{aligned}
$$

omitting the time argument of $f$ to simplify notation.
In the following, we use

$$
f_kf_l - \hat{f}_k\hat{f}_l = f_k(f_l - \hat{f}_l) + \hat{f}_l(f_k - \hat{f}_k)
$$

together with boundedness of shift operator $(Sf)_k = f_{k+1}$, i.e. $\|Sf\|_2^2 \leq \|f\|_2^2$, and the Cauchy-Schwarz inequality

$$
\langle g, h\rangle := \sum_{k\geq 0} g_kh_k \leq \|g\|_2\|h\|_2 \quad \text{for all } g,h \in \ell^2.
$$

Hence, we get

$$
\begin{aligned}
\frac{d}{dt}\|f-\hat{f}\|_2^2 &= 2\Bigg[\sum_{k,l\geq 1}(f_k-\hat{f}_k)\,c(l,k-1)\Big(f_{k-1}(f_l-\hat{f}_l) + (f_{k-1}-\hat{f}_{k-1})\hat{f}_l\Big) \\
&\quad + \sum_{k,l\geq 0}(f_k-\hat{f}_k)\,c(k+1,l)\Big(f_{k+1}(f_l-\hat{f}_l) + (f_{k+1}-\hat{f}_{k+1})\hat{f}_l\Big) \\
&\quad + \sum_{k,l\geq 0}(f_k-\hat{f}_k)(c(k,l)+c(l,k))\Big(f_k(f_l-\hat{f}_l) + (f_k-\hat{f}_k)\hat{f}_l\Big)\Bigg] \\
&\leq 2\Big[\langle S|f-\hat{f}|, \hat{Q}_f|f-\hat{f}|\rangle + \bar{C}\|f-\hat{f}\|_2^2 \\
&\quad + \langle |f-\hat{f}|, SQ_f|f-\hat{f}|\rangle + \bar{C}\langle |f-\hat{f}|, S|f-\hat{f}|\rangle \\
&\quad + \langle |f-\hat{f}|, (Q_f+\hat{Q}_f)|f-\hat{f}|\rangle + 2\bar{C}\|f-\hat{f}\|_2^2\Big] \\
&\leq 16\,\bar{C}\|f-\hat{f}\|_2^2,
\end{aligned}
$$

having also used $f_k, \hat{f}_k \leq 1$ for all $k \geq 0$ and (3.26) to get the third inequality.

Since we assume the initial condition $f(0) - \hat{f}(0) = 0$, by Gronwall's inequality we get

$$\|f(t)-\hat{f}(t)\|_2^2 \leq \|f(0)-\hat{f}(0)\|_2^2 \exp(16\,\bar{C}t) = 0 \quad \text{for all } t \geq 0.$$

Hence, $f(t) = \hat{f}(t)$ for all $t \geq 0$ and the solution $t \mapsto f(t)$ is unique. $\square$

### 3.3.4 Proof of Theorem 3.2

*Proof.* We use equivalence of propagation of chaos with a law of large numbers for the empirical measures, as explained in Appendix A. Recall,

$$f_k^L(t) = \mathbb{E}^L[F_k(\eta(t))].$$

We have

$$F_k^2(\eta(t)) = \frac{1}{L^2} \sum_{x,y \in \Lambda} \delta_{\eta_x,k} \delta_{\eta_y,k},$$

and by taking the expectation, we get

$$\begin{aligned}
\mathbb{E}[F_k^2(\eta(t))] &= \frac{1}{L^2} \sum_{x,y \in \Lambda} \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = k] \\
&= \mathbb{P}[\eta_x(t) = k, \eta_y(t) = k] + O(1/L), \quad \text{where } x \neq y \\
&\to f_k^2(t) \quad \text{as } L \to \infty.
\end{aligned}$$

This follows in full analogy with the derivation after (3.24).
This also implies $\mathrm{Var}^L[F_k(\eta(t))] \to 0$ as $L \to \infty$ and a weak law of large numbers $F_k(\eta(t)) \to f_k(t)$ for all $t \geq 0$. $\square$

## 3.4 Properties of solutions

### 3.4.1 Conserved quantities

Since $f_k(t)$ are limits of $f_k^L(t) \in [0, 1]$, we have $f_k(t) \in [0, 1]$ for all $k \geq 0, t \geq 0$. We denote the $i^{th}$ moment of $f(t)$ by

$$m_i(t) = \sum_{k \geq 0} k^i f_k(t).$$

The limiting mean-field equation (3.9) is the master equation of the non-linear birth death chain $(Y_t : t \geq 0)$ on $\mathbb{N}_0$ so that $f_k(t) = P[Y_t = k]$ with generator

$$\mathcal{L}_{\mathrm{BD}} h(k) = \sum_{l\geq 0} c(k,l) f_l(t)(h(k-1) - h(k)) + \sum_{l\geq 0} c(l,k) f_l(t)(h(k+1) - h(k)), \quad (3.27)$$

where $c(0,l) = 0$ for all $l \geq 0$. This is the limit dynamics of the single site process (3.8), and the time dependent birth rates $\beta_k(t)$ and death rates $\mu_k(t)$ are given by

$$\beta_k(t) = \sum_{l\geq 0} c(l,k) f_l(t) \quad \text{and} \quad \mu_k(t) = \sum_{l\geq 0} c(k,l) f_l(t). \quad (3.28)$$

Note that this immediately implies that $f_k = \delta_{0,k}$ is stationary, but in general 0 is not an absorbing state as long as $f_k(0) > 0$ for some $k > 0$, as discussed in detail later. The adjoint operator $\mathcal{L}_{\mathrm{BD}}^{\dagger}$ then characterises the right-hand side of the master equation (3.9) which can be written as

$$\frac{d}{dt} f(t) = \mathcal{L}_{\mathrm{BD}}^{\dagger} f(t).$$

$f(t)$ is indeed a probability distribution on $\mathbb{N}_0$ for all $t \geq 0$ since we have

$$\mathcal{L}_{\mathrm{BD}} 1 = 0 \quad \text{and therefore} \quad m_0(t) = m_0(0) = 1.$$

Also, as usual $\mathcal{L}_{\mathrm{BD}} k = \beta_k(t) - \mu_k(t)$, which leads with (3.28) to

$$\frac{d}{dt} m_1(t) = \sum_{k\geq 0} f_k(t) \mathcal{L}_{\mathrm{BD}} k = \sum_{k\geq 0} \sum_{l\geq 0} f_k(t) f_l(t) \big(c(l,k) - c(k,l)\big) = 0.$$

This implies that the expectation is conserved for the chain $(Y_t : t \geq 0)$, i.e.

$$m_1(t) = m_1(0) =: \rho \geq 0,$$

which corresponds to the particle density $\rho$ in the original particle system.
Note, however, that $(Y_t : t \geq 0)$ is not a martingale since $\mathcal{L}_{\mathrm{BD}} k \neq 0$, and the conservation of $m_1$ results from the non-linearity of the process. By assumption (2.25) on the rates $c$, the chain is further irreducible unless $f(0)$ is degenerate, but we will see below that the additional conserved quantity leads to non-uniqueness for the stationary distribution.

### 3.4.2 Stationary distributions

Under condition (2.27) on the rates, we know that for each $\phi \in D_\phi$, the single site marginal of the grand canonical product measures $\nu_\phi$

$$f_k^\phi := \nu_\phi[\eta_x = k]$$

is a stationary solution of (3.9). From the stationary marginal of the process (2.30), we have the relation

$$\frac{c(k,0)}{c(1,k-1)} f_k^\phi = \phi f_{k-1}^\phi \quad \text{for all } k \geq 1 \text{ and } \phi \in D_\phi. \tag{3.29}$$

With the usual convention $f_{-1}^\phi = 0$ and $c(0,l) = 0$ for all $l \geq 0$ this leads to

$$\begin{aligned}
\mathcal{L}_{\mathrm{BD}}^\dagger f_k^\phi &= \sum_{l\geq 0} c(k+1,l) f_l^\phi f_{k+1}^\phi + \sum_{l\geq 1} c(l,k-1) f_l^\phi f_{k-1}^\phi \\
&\quad - \Big( \sum_{l\geq 1} c(l,k) f_l^\phi + \sum_{l\geq 0} c(k,l) f_l^\phi \Big) g f_k^\phi \\
&= \sum_{l\geq 1} c(k+1,l-1) f_{l-1}^\phi \frac{c(1,k)}{c(k+1,0)} \phi f_k^\phi + \sum_{l\geq 0} c(l+1,k-1) f_{l+1}^\phi \frac{c(k,0)}{c(1,k-1)} \frac{1}{\phi} f_k^\phi \\
&\quad - \Big( \sum_{l\geq 1} c(l,k) f_l^\phi + \sum_{l\geq 0} c(k,l) f_l^\phi \Big) f_k^\phi \\
&= \sum_{l\geq 1} c(l,k) \frac{c(1,l-1)}{c(l,0)} \phi f_{l-1}^\phi f_k^\phi + \sum_{l\geq 0} c(k,l) \frac{c(l+1,0)}{c(1,l)} \frac{1}{\phi} f_{l+1}^\phi f_k^\phi \\
&\quad - \Big( \sum_{l\geq 1} c(l,k) f_l^\phi + \sum_{l\geq 0} c(k,l) f_l^\phi \Big) f_k^\phi \\
&= 0,
\end{aligned}$$

where in the second equality and the last equality we use (3.29) and in the third equality we use condition (2.27) of the rate. Therefore, under condition (2.27), we have an explicit stationary distribution for each value $\rho = m_1(0)$ of the conserved first moment which is not larger than $\rho_c$ (see Definition 2.12), given by $f^\phi$ with $\phi \in D_\phi$ such that $R(\phi) = \rho$.

### 3.4.3 Initial conditions and ergodic behaviour

Consider a fixed initial condition $f(0)$ for the limit equation (3.9) with finite density $\rho = m_1(0) \in (0, \infty)$. A natural corresponding sequence of initial conditions for the particle system are simply product measures $\nu^L$ with marginals $\nu^L[\eta_x = \cdot] = f(0)$, in which case $f^L(0) = f(0)$ for all $L \geq 1$. Another useful choice is a conditional version of these measures with a fixed number of particles

$$\pi_{L,N} = \nu^L\Big[\, \cdot \,\Big|\sum_{x\in\Lambda} \eta_x = N\Big] \text{ and } f^L(0) = \pi_{L,N}[\eta_x = \cdot]. \tag{3.30}$$

If $N$ is chosen to increase with $L$ such that $N/L \to \rho$, then $f^L(0) \to f(0)$ as $L \to \infty$ weakly and in total variation distance. The formulation of our main result requires i.i.d. initial conditions (3.4), which provide permutation invariance of the dynamics and are otherwise used only in (3.25). Permutation invariance is also given under the conditional measures (3.30), and the condition introduces only a small negative correlation between different occupation numbers $\eta_x(0)$ and $\eta_y(0)$ of order $1/L$. This leads to a vanishing correction in (3.25), and the proof can be easily adapted to also cover initial conditions with a fixed number of particles.

A generic initial condition of the form (3.30) is to simply distribute $N$ particles uniformly at random, leading to binomial marginals

$$\binom{N}{k}\left(\frac{1}{L}\right)^k\left(1-\frac{1}{L}\right)^{N-k} = f_k^L(0) \to f_k(0) = \frac{\rho^k}{k!}e^{-\rho}, \tag{3.31}$$

converging to $\mathrm{Poi}(\rho)$ variables as $N, L \to \infty$ and $N/L \to \rho$.

Given a family of stationary measures $f^\phi$, a natural question is that of ergodicity, i.e. for initial conditions $f(0)$ with first moment $\rho = m_1(0) < \infty$, does $f(t)$ converge to $f^\phi$ with $R(\phi) = \rho$? While contraction arguments may by possible for particular jump rates $c(k,l)$, we are not aware of general results on convergence to stationary solutions for non-linear dynamical systems that would answer this question. As discussed in Section 2.2.1, on the restricted state space $\big\{\eta \in E : \sum_{x\in\Lambda} \eta_x = N\big\}$ the process $(\eta(t) : t \geq 0)$ is a finite state, irreducible Markov chain, which converges to its unique stationary distribution $\pi_{L,N}$. The equivalence of ensembles ensures weak convergence

$$\pi_{L,N}[\eta_x = k] \to f_k^\phi \quad \text{as } L, N \to \infty,\ N/L \to \rho \tag{3.32}$$

provided that $R(\phi) = \rho \leq \rho_c$. For condensing models with $\rho > \rho_c$, the above holds with $\phi = \phi_c < \infty$, which corresponds to a loss of mass in the condensate since the limit has only first moment $\rho_c < \rho$. Due to ergodicity for a finite state Markov chain, we have

$$f^L(t) \to \pi_{L,N}[\eta_x = \cdot] \quad \text{as } t \to \infty$$

for each finite $L$, which holds in total variation or $L^2$ distance.

If the convergence $f^L(t) \to f(t)$ was uniformly in $t > 0$, this could be used to establish ergodicity for the limit process. However, the error bounds arising from Lemma 3.2 are in fact of order $e^{Ct}/L$ for some $C > 0$, since the branching processes (3.18) in our coupling argument grow exponentially in time. They are clearly only useful for $t \ll \log L$ (in particular for all fixed $t > 0$), and our proof does not provide uniform convergence. In fact, ergodicity breaking is a well-known phenomenon in the presence of phase transitions, e.g. for the contact process uniqueness of the stationary distribution is lost in infinite volume. For solutions to (3.9), however, we still expect ergodicity at least for $\rho \leq \rho_c$, and explicit heuristic scaling solutions for particular systems discussed in the next section support this even for $\rho > \rho_c$.

Note that our main result in Theorem 3.1 holds independently of condition (2.27) and instead requires boundedness of the rates $c$. Without condition (2.27) we still expect a continuous family of stationary distributions for the birth death chain indexed by the first moment with similar ergodicity properties, but we are not aware of related results. Results on some particular cases of non-linear birth death chains can be found in [37] where the chain has normal state-dependent death rates but birth rates are in the form of a time-dependent mean.

## 3.5 Discussion

We have established the mean-field equation (3.9) as the limit dynamics of stochastic particle systems, which provides an important ingredient for a rigorous analysis of the coarsening dynamics of condensing stochastic particle systems. Our result holds under arguably quite restrictive conditions, which we discuss in detail in the following.

- Theorem 3.1 is formulated for i.i.d. initial conditions (3.4), and we have discussed in Section 3.4 how this can be extended to conditional product measures which introduce vanishing correlations and are permutation invariant. In our proof, permutation invariance is only used to establish existence in Section 3.3.1. This makes use of (3.6) implying that the single site process $\eta_x(t)$ provides a realisation of the limiting birth death chain. Since all estimates in Section 3.3.1 hold uniformly in $x$, a similar argument can be used to establish tightness for the empirical process $(F_k(\eta_t) : k \geq 0)$. This would allow for non-permutation invariant initial conditions with vanishing correlations and a result on convergence of $f_k^L(t)$. Its limit could then, however, not be interpreted as the single site distribution.

- Mean-field equations (3.9) are often used as approximations in other geometries such as symmetric or asymmetric dynamics on $d$-dimensional regular lattices. As usual, the larger the dimension the better the approximation, see e.g. [48, 21, 35] for details. Since our result does not involve any time scaling, mean-field averaging of the birth and death rates (3.28) is achieved by a diverging number of neighbours of each lattice site. This is a crucial ingredient in our proof in (3.22) and in fact essential for any rigorous derivation of (3.9). Our arguments could be directly extended to graphs which are not complete but have a weaker version of the above property.

- Condensing stochastic particle systems exhibit several time scales diverging with the system size (e.g. for ZRPs, this has been studied in [48]), some of which have been identified recently also on a rigorous basis including hydrodynamics [91] and also metastable dynamics of the condensate [11, 5]. As we discussed in Section 3.4, convergence in our result does not hold uniformly in time, and error estimates vanish on time scales at most of order $\log L$ due to the coupling with branching processes. Time scales related to possible condensation dynamics are typically at least polynomial in $L$ also on complete graphs (see [48] for ZRP). Therefore, our result only describes the initial part of the dynamics which still contains interesting coarsening phenomena as seen in the next chapter.

- Boundedness of jump rates (3.7) is the most restrictive condition that we expect to be not necessary for the limit result to hold, but which would require a significant extension of our proof including e.g. a priori bounds on occupation numbers to use cut-off arguments. Alternatively, one can try to adapt a proof in [84] for interacting diffusions, which only requires Lipchitz continuity in the rates. This would, however, also not cover for the inclusion process, for which our result can be established with other techniques as discussed in Chapter 5.

# Chapter 4

# Coarsening dynamics in condensing zero-range processes and size-biased birth death chains

## 4.1 Introduction

In Chapter 3, we presented a rigorous derivation of the mean-field equation for the single site dynamics of misanthrope-type processes on a complete graph. In this chapter, we apply that in a specific model in which the jump rates only depend on the number of particles in the departure site, namely the zero-range process as introduced in Section 2.2.3. Based on the mean-field master equation, we analyse the coarsening dynamics of the single site process $f_k$ using the method studied in [44, 46] for zero-range processes with the generator

$$(\mathcal{L}_{\mathrm{ZRP}}h)(\eta) = \sum_{x,y\in\Lambda_L} g(\eta_x)q(x,y)(h(\eta^{x\to y}) - h(\eta)), \qquad \text{(2.38 revisited)}$$

with jump rates of the form

$$g(k) = \begin{cases} 0 & \text{if} \quad k = 0, \\ 1 + \frac{b}{k^\gamma} & \text{otherwise,} \end{cases} \qquad \text{(2.41 revisited)}$$

for any constant $b > 0$ and $\gamma \in (0, 1]$. We then study the coarsening scaling law for values of the parameter $\gamma \in (0, 1]$. In addition to extending the results in [44] to a larger class of parameter values (which have also been studied in [46]), the main novelty is a direct description of the size-biased version of the single site dynamics. It provides an interesting tool to analyse the coarsening dynamics without any finite size effects and is also numerically more efficient than previous approaches. The volume of the condensed phase does not vanish in time as in the zero-range process or the direct mapping to birth death chains as presented in Chapter 3. As usual, we focus the presentation of our results on processes on complete graphs i.e. $q(x, y) = 1/(L-1)$ for all $x \neq y$ with finite lattices $\Lambda$ of size $L$, and discuss the role of other homogeneous geometries in Section 4.4.

This chapter is organised as follows. In Section 4.2, we introduce and analyse the dynamics of site empirical process $f_k$ and its size-biased version $p_k$. This is followed by Section 4.3 in which we provide details for the implementation of non-linear birth death chains and give results which are corroborated by simulation data. We then discuss and conclude in Section 4.4.

The results of this chapter have been published in [60], and the scaling analysis for $\gamma \in (0, 1)$ has been done in more detail afterwards in [46].

## 4.2 Dynamics of the process

### 4.2.1 Empirical processes

Consider a zero-range process (2.38) with bounded rates (2.41) on a finite lattice of size $L$ with $N$ particles, initially distributed uniformly at random. To describe the coarsening dynamics, we will study dynamics of two kinds of empirical processes:

$$\text{Site empirical process} \quad F_k(\eta(t)) := \frac{1}{L} \sum_{x \in \Lambda} \delta_{\eta_x(t),k}, \qquad \text{(3.1 revisited)}$$

$$\text{Size-biased empirical process} \quad P_k(\eta(t)) := \frac{1}{N} \sum_{x \in \Lambda} k \delta_{\eta_x(t),k} \ . \qquad (4.1)$$

The first process is defined in Chapter 3 as (3.1), while the second one counts the fraction of particles which are on sites with occupation number $k$. Note that the condensed phase has only a small contribution in the first process which decreases in time and tends to the stationary value $1/L$ with only one condensate site. In the second process, however, the weight of the condensed phase is roughly constant in time and given by the mass fraction $(\rho - \rho_c)/\rho$ due to the different weighting, where $N/L \to \rho$ is the limiting particle density. This is, therefore, more useful to describe

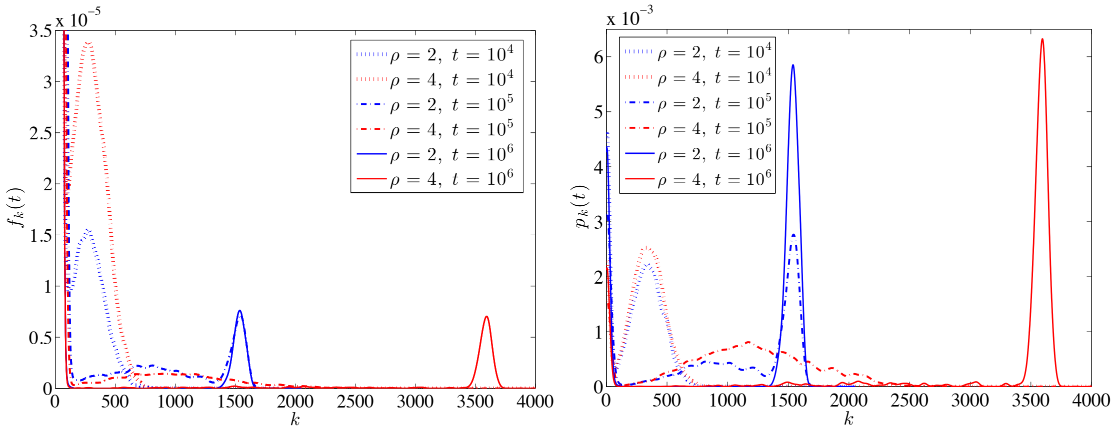



Figure 4.1: Time evolution of the mass distribution as described by the empirical processes (3.1) and (4.1) for a system with $L = 1024$, $\gamma = 1$, $b = 4$ and two different densities $\rho = 2$ and 4, which are larger than $\rho_c = 0.5$. Expectations $f_k(t)$ (left) and $p_k(t)$ (right) are defined in (3.2) and (4.19). Outputs are at three different times, the first two during the coarsening regime, the last one when stationarity is reached. Note that the different scales of the $y$-axes indicate the larger weight of the condensed phase under the $P_k$ process. The bulk part of the distribution concentrates on the far left and is not resolved in this plot. Data are averaged over 500 realisations.

the dynamics in the condensed phase. Both processes are normalized, i.e.

$$\sum_{k=0}^{\infty} F_k(\eta) = \sum_{k=1}^{\infty} P_k(\eta) = 1 \text{ for all } \eta \in E_{L,N}.$$

Also, they can be interpreted as distributions of single site occupations. The second process is a size-biased version of the first one, and both are related via

$$LkF_k(\eta) = NP_k(\eta) \quad \text{for all } \eta \in E_{L,N} \quad \text{and } k \geq 1. \tag{4.2}$$

In Figure 4.1, we illustrate the behaviour of these processes when averaged over 500 realisations of the zero-range dynamics. We see that the coarsening process transports mass from small occupation numbers to form a stationary bump around occupation numbers $(\rho - \rho_c)L$, two such bumps are shown for densities $\rho = 2$ and 4. The distribution within the condensed phase at any given time is characterised by an intermediate bump which is broadening over time with its maximum moving to the right. Building on the analysis in [44], we investigate the scaling behaviour of this bump. On a finite lattice, the scaling behaviour only occurs in a finite time window before the system reaches stationarity. However, ideally, we are interested in an infinite lattice where the condensed phase distribution keeps broadening and

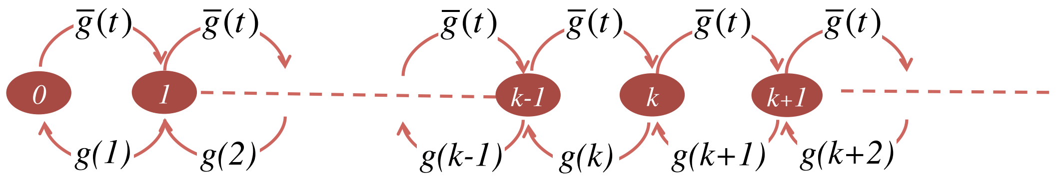



Figure 4.2: Diagram of the ZRP birth death chain $Y_t$.

moving to larger occupation numbers forever. Both processes in Figure 4.1 are shown on very different scales since the condensed phase has much larger weight under $P_k$ than under $F_k$, as explained above. The bulk part of the distribution concentrates on the left and is not shown in full since we focus our attention on the condensed phase.

### 4.2.2 Analysis of $F_k(\eta)$

While the size-biased process $P_k$ is more suitable to study the coarsening dynamics, we start our analysis with the process $F_k$ describing the single site dynamics of the process since this is simpler to analyze. This has already been studied in [44] for $\gamma = 1$, so we keep the presentation short and show numerical results only for the size-biased process later. For our zero-range processes assumptions in Theorem 3.1 are satisfied which leads to the mean-field equation (3.9) taking the form

$$\frac{df_k(t)}{dt} = g(k+1)f_{k+1}(t) + \bar{g}(t)f_{k-1}(t) - (g(k) + \bar{g}(t))f_k(t), \tag{4.3}$$

valid for all $k \geq 0$ with the convention $f_{-1}(t) \equiv 0$. The short hand

$$\bar{g}(t) = \sum_k g(k)f_k(t) \tag{4.4}$$

indicates the time-dependent expectation of the jump rate. As before, this is the master equation of a birth death chain with $k$-independent birth rate $\bar{g}(t) = \sum_k g(k)f_k(t)$ and time-independent but state dependent death rate $g(k)$, which have been studied in [37]. In the following, we denote the corresponding process by $(Y_t : t \geq 0)$ on state space $\mathbb{N}_0$, so that $f_k(t) = P[Y_t = k]$. Note that the birth rate is itself an expectation with respect to $f_k(t)$ (4.4), so that this is still a non-linear master equation. This chain describes the single site dynamics of the zero-range process. With our choice of $\bar{g}(t)$ in (4.4) we see that

$$\begin{aligned} \frac{d}{dt}\sum_k k f_k(t) &= \sum_k g(k+1) k f_{k+1}(t) + \bar{g}(t) \sum_k k f_{k-1}(t) \\ &\quad - \sum_k g(k) k f_k(t) - \bar{g}(t) \sum_k k f_k \\ &= -\sum_k g(k) f_k(t) + \bar{g}(t) = 0 \ , \end{aligned} \tag{4.5}$$

which is consistent with mass conservation and $\sum_k k f_k(t) = \rho$ for all $t \geq 0$.
Due to this additional conserved quantity, the non-linear chain has a whole family of stationary distributions, and indeed it is easy to see that the marginals

$$f_k^{\phi} := \nu_\phi[\eta_x = k] = \frac{1}{z(\phi)} \frac{\phi^k}{g!(k)}$$

with weight (2.39) of the zero-range process are stationary for all $\phi \in [0,1]$. For every initial distribution with density $\rho \leq \rho_c$, the chain should converge to the corresponding distribution with $R(\phi) = \rho$. For supercritical densities $\rho > \rho_c$, we expect the chain to converge to $f^1$ in analogy to the behaviour of the zero-range process summarised in Section 2.2.1. The solution of (4.3) will develop a bimodal structure over time, corresponding to the bulk and the condensed part of the distribution. This separation is characterised by a scaling parameter $\epsilon_t \to 0$ as $t \to \infty$, denoting the volume fraction of the condensed phase. Following the approach in [44], we therefore make the ansatz for a separated state

$$f_k(t) = \underbrace{f_k(t)\, \mathbb{I}_{[0,1/\sqrt{\epsilon_t}]}(k)}_{:=f_k^{\text{bulk}}(t)} + \underbrace{f_k(t)\, \mathbb{I}_{(1/\sqrt{\epsilon_t},\infty)}(k)}_{:=f_k^{\text{cond}}(t)} \ . \tag{4.6}$$

The split of the two contributions at occupation numbers $k = O(\epsilon_t^{-1/2})$ is arbitrary, both phases are clearly separated as can be seen later in Figure 4.6, and any other power in $(-1,0)$ would work equally well. The bulk part of the distribution converges, i.e.

$$f_k^{\text{bulk}}(t) \to f_k^1 = \nu_1[\eta_x = k] \quad \text{as } t \to \infty, \tag{4.7}$$

and in fact is very close to the stationary marginal $\nu_1$ even for finite times as is confirmed later in Figure 4.5. The probability fraction of this part is $(1 - \epsilon_t) \to 1$ and the expectation converges to $\rho_c$ corresponding to the mass contained in the bulk. The condensed part therefore contains the rest of the mass $\rho - \rho_c$ according to (4.5), but only a vanishing probability of order $\epsilon_t$. Therefore, typical occupation numbers in this phase scale as $1/\epsilon_t$, which justifies the intermediate scale chosen in the formal Definition (4.6). Furthermore, we assume that $f_k^{\text{cond}}(t)$ takes the scaling form

$$f_k^{\text{cond}}(t) = \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t \text{ and } \epsilon_t = t^{-\frac{1}{\gamma+1}} \ , \tag{4.8}$$

which we adapted from [44] to $\gamma \in (0, 1]$. One of the prefactors $\epsilon_t$ corresponds to the vanishing probability of the condensed phase, the other one is a volume element to get a density $h(u)$. In terms of the scaling function $h(u)$, the volume fraction of the condensed phase is given by

$$\sum_k f_k^{\text{cond}}(t) = \sum_{k \geq 1/\sqrt{\epsilon_t}} \epsilon_t^2 h(k\epsilon_t) \approx \epsilon_t \int_0^\infty h(u) du = O(\epsilon_t). \tag{4.9}$$

Also, the mass in the condensed phase is

$$\sum_k k f_k^{\text{cond}}(t) = \sum_{k \geq 1/\sqrt{\epsilon_t}} \epsilon_t^2 k h(k\epsilon_t) \approx \int_0^\infty u h(u) du = \rho - \rho_c, \tag{4.10}$$

which fixes the normalization of $h$ consistently with previous assumptions. Since the bulk phase is close to the stationary marginal $f^1$ and $h(u)$ changes with time only on scale $\epsilon_t$, we have

$$\frac{d}{dt} f_k^{\text{bulk}}(t) = O(\dot{\epsilon_t}) \ll O(\epsilon_t) = \frac{d}{dt} f_k^{\text{cond}}(t) \ , \tag{4.11}$$

as we will see later that $\epsilon_t$ decays as a power law with $t$. Therefore, the condensed part fulfills the same master equation as (4.3)

$$\frac{d}{dt} f_k^{\text{cond}}(t) = g(k+1) f_{k+1}^{\text{cond}}(t) + \bar{g}(t) f_{k-1}^{\text{cond}}(t) - (g(k) + \bar{g}(t)) f_k^{\text{cond}}(t). \tag{4.12}$$

To write this as a closed equation for $h(u)$, we need to find an expression for the time-dependent birth rate $\bar{g}(t) = \sum_{k \geq 1} g(k) f_k(t)$ under the phase separated state. In the canonical ensemble, the mobility (2.40) for ZRP is simply given by a ratio of the partition functions

$$\chi_{L,N} := \mathbb{E}_{\pi_{L,N}}[g(\eta_x)] = Z_{L,N-1}/Z_{L,N}. \tag{4.13}$$

This can be easily computed numerically using recursions of the form

$$Z_{L,N} = \sum_{n=0}^{N} w(n) Z_{L-1,n},$$

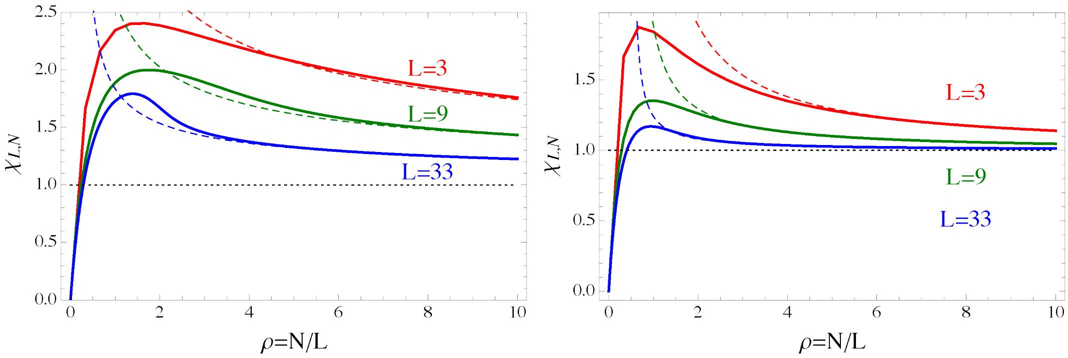



Figure 4.3: Approximation of the canonical mobility (4.13) (bold line) by the jump rates (dashed line) as predicted in (4.14). Parameter values are $b = 4$, $\gamma = 0.5$ (left) and $\gamma = 1$ (right).

and it is known [23] that the supercritical mobility for $\rho > \rho_c$ is well approximated by the jump rates of the condensate as

$$\chi_{L,\rho L} \approx g((\rho - \rho_c)L) = 1 + \frac{b}{(\rho - \rho_c)^\gamma L^\gamma}. \tag{4.14}$$

This is illustrated in Figure 4.3, and this approximation works well even for very small system sizes. Now assume that $1/\epsilon_t$ is the typical separation between cluster sites, and the system is locally stationary in between clusters. The main assumption is that the phase separated state in the limit of diverging system size is given by

$$f_k(t) = \pi_{1/\epsilon_t, \rho/\epsilon_t},$$

i.e. the canonical, super-critical state with time-dependent system size corresponding to the typical distance between clusters. So we can replace $L$ by $1/\epsilon_t$ in (4.14) to get the prediction

$$\bar{g}(t) = 1 + \epsilon_t^\gamma \frac{b}{(\rho - \rho_c)^\gamma},$$

which we then write in the form

$$\bar{g}(t) = 1 + A\epsilon_t^\gamma\ , \tag{4.15}$$

where for $\rho$ large enough we expect $A = b/(\rho - \rho_c)^\gamma$. Plugging this into (4.8) and (4.12), we get for the leading order terms

$$\frac{\dot{\epsilon}_t}{\epsilon_t}[uh'(u) + 2h(u)] = \epsilon_t^{\gamma+1}\left(\epsilon_t^{1-\gamma}h''(u) + \frac{b}{u^\gamma}h'(u) - Ah'(u) - \frac{b\gamma}{u^{\gamma+1}}h(u)\right). \tag{4.16}$$

Equating powers of $\epsilon_t$ on both sides leads to the choice $\epsilon_t = t^{-\frac{1}{\gamma+1}}$ and the equation

$$t^{-\frac{1-\gamma}{1+\gamma}} h''(u) + \left(\frac{u}{(\gamma+1)} + \frac{b}{u^\gamma} - A\right) h'(u) + \left(\frac{2}{(\gamma+1)} - \frac{b\gamma}{u^{\gamma+1}}\right) h(u) = 0, \quad (4.17)$$

for the scaling function $h$. For $\gamma = 1$, we simply have

$$h''(u) + \left(\frac{1}{2}u - A + \frac{b}{u}\right) h'(u) + \left(1 - \frac{b}{u^2}\right) h(u) = 0, \quad (4.18)$$

as was derived in [44] which is indeed time-independent. For $\gamma < 1$ we have to keep a small time-dependent prefactor for $h''$ to regularize the solutions. This is necessary since we impose conditions on the first moment on $h$ as given in (4.10) and also require $h(u) \to 0$ as $u \to \infty$. A more detailed discussion of this point can be found in [46]. Neither equation (4.17) nor (4.18) can be solved explicitly, but numerical solutions presented in the next section agree well with simulation data.

### 4.2.3 Analysis of $P_k(\eta)$

Analogously to $F_k$, we can act with the generator (2.38) on the test function $P_k$ (4.1). In this section, we would like to apply our result from Theorem 3.1 for the size-biased process (4.1), and denote

$$p_k^L(t) = \mathbb{E}^L[P_k(\eta(t))]. \quad (4.19)$$

Using (4.2), we can get convergence of $p_k^L(t)$ by

$$p_k^L(t) = \mathbb{E}\left[\frac{kL}{N} F_k(\eta(t))\right] \to p_k(t) := \frac{k}{\rho} f_k(t), \quad (4.20)$$

for all $k \geq 0$ and $t \geq 0$. This implies that the size-biased process converges weakly to another kind of birth death chain with distribution $p(t) = (p_k(t) : k \in \mathbb{N})$. In this case, however, the mean-filed master equations are more complicated. Using (4.3) and (4.19), we get for all $k > 1$

$$\frac{dp_k(t)}{dt} = \frac{k}{\rho}\left[g(k+1)\frac{\rho}{k+1}p_{k+1}(t) + \bar{g}(t)\frac{\rho}{k-1}p_{k-1}(t) - (g(k) + \bar{g}(t))\frac{\rho}{k}p_k(t)\right].$$

For $k = 1$, we have

$$\begin{aligned}\frac{d}{dt}p_1(t) &= -g(1)p_1(t) - \bar{g}(t)p_1(t) + \frac{1}{\rho}\bar{g}(t)f_0(t) + \frac{1}{2}g(2)p_2(t)\\ &= \frac{1}{2}g(2)p_2(t) - 2\bar{g}(t)p_1(t) + \sum_{k\geq 2}\frac{1}{k}(g(k)-\bar{g}(t))p_k(t),\end{aligned} \tag{4.21}$$

where we used (4.4) and that $f_0(t) = 1 - \sum_{k=1}^{\infty} f_k(t) = 1 - \rho\sum_{k=1}^{\infty}\frac{p_k(t)}{k}$.
For $k > 1$, we can write

$$\begin{aligned}\frac{d}{dt}p_k(t) &= -g(k)p_k(t) - \bar{g}(t)p_k(t) + \frac{k}{k-1}\bar{g}(t)p_{k-1}(t) + \frac{k}{k+1}g(k+1)p_{k+1}(t)\\ &= \frac{k}{k+1}g(k+1)p_{k+1}(t) + \frac{k}{k-1}\bar{g}(t)p_{k-1}(t)\\ &\quad - \left(\frac{k-1}{k}g(k) + \frac{k+1}{k}\bar{g}(t)\right)p_k(t) + \frac{1}{k}(\bar{g}(t)-g(k))p_k(t).\end{aligned} \tag{4.22}$$

As before, this can be thought of as the master equation of a birth death chain on the state space $\mathbb{N}$, with additional long-range jumps from occupation numbers $k > 1$ to $k = 1$, and diagonal terms corresponding to killing or cloning events which are not probability conserving. We have

$$\begin{aligned} &\text{birth rate} && \frac{k+1}{k}\bar{g}(t), \ \text{ for } k > 0\ ,\\ &\text{death rate} && \frac{k-1}{k}g(k), \ \text{ for } k > 1\ ,\\ &\text{rate from } k \text{ to } 1 && \frac{1}{k}(g(k)-\bar{g}(t))_+, \ \text{ for } k > 1\ ,\\ &\text{cloning rate} && \frac{1}{k}(\bar{g}(t)-g(k))_+, \ \text{ for } k > 1\ ,\\ &\text{killing rate} && \sum_{k>1}\frac{1}{k}(\bar{g}(t)-g(k))_+, \ \text{ for } k = 1\ ,\end{aligned} \tag{4.23}$$

where we denote by $(\cdot)_+ = \max\{0, (\cdot)\}$ the positive part of the expression. Note that the total cloning rate of chains with $k > 1$ exactly equals the killing rate of chains with $k = 1$, so in total the probability is conserved

$$\frac{d}{dt}\sum_{k\geq 1}p_k(t) = \sum_{k\geq 2}\frac{p_k(t)}{k}(\bar{g}(t)-g(k)) + \sum_{k\geq 2}\frac{p_k(t)}{k}(g(k)-\bar{g}(t)) = 0.$$

Using (4.2), the average jump rate is now given by

$$\bar{g}(t) = \rho \sum_{k \geq 1} \frac{g(k)}{k} p_k(t). \tag{4.24}$$

If $\bar{g}(t) < 1$, all cloning and killing rates vanish and the only new part of the dynamics are long range jumps from $k$ to 1. This leads to subcritical dynamics, and it is easy to see that now the size-biased version

$$p_k^\phi := \bar{\nu}_\phi[\eta_x = k] := \frac{k}{R(\phi)} \nu_\phi[\eta_x = k] \tag{4.25}$$

of the marginals of stationary measures of the zero-range process (2.30) with weight (2.39) are stationary for the birth death chain with master equation (4.22) for all $\phi \in [0, 1]$. Note also that there is no obvious second conservation law for the size-biased chains related to the density as was the case for $f_k(t)$. However, $\rho$ now explicitly enters the master equation of the process through the above expression for $\bar{g}(t)$ in (4.24), which selects the stationary distribution for different $\rho \in [0, \rho_c]$. Any long jumps from $k$ to 1 can be interpreted as sites with $\eta_x = 0$ receiving a particle in the original zero-range processes. For supercritical systems with $\bar{g}(t) \geq 1$, chains with small occupation number $k$ perform jumps to 1 since $g(k) > \bar{g}(t)$, while chains that made it to large occupation numbers do not, but have a positive rate for cloning. This mechanism generates bimodal distributions with a condensed and a bulk phase denoted by $p_k^{\mathrm{cond}}$ and $p_k^{\mathrm{bulk}}$, analogously to (4.6). Using the same scaling ansatz as (4.6) where we replace the asymptotic bulk part with a size-biased version, we note that $\rho p_k(t) = k f_k(t)$ implies the same relation for the condensate part of the distribution, i.e. $\rho p_k^{\mathrm{cond}}(t) = k f_k^{\mathrm{cond}}(t)$. This leads to

$$\sum_k p_k^{\mathrm{cond}}(t) = \frac{1}{\rho} \sum_k k f_k^{\mathrm{cond}}(t) = \frac{\rho - \rho_c}{\rho}$$

for the mass fraction in the condensed phase which does not vanish and is constant in time. In particular, for the scaling of $p_k^{\mathrm{cond}}(t)$, we have

$$p_k^{\mathrm{cond}}(t) = \frac{1}{\rho} k f_k^{\mathrm{cond}}(t) = \frac{1}{\rho} u h(u) \epsilon_t. \tag{4.26}$$

Therefore, it is sufficient to solve the equations (4.17) or (4.18) to get theoretical predictions for the scaling behaviour of $p_k^{\mathrm{cond}}(t)$.

## 4.3 Main Results for the coarsening dynamics

### 4.3.1 Implementation of non-linear birth death chains

Simulation of the birth death chains related to $f_k$ and $p_k$ can only be done approximately using a large ensemble of parallel realisations, due to the non-linearity of the master equations. To determine the time evolution, it is necessary to compute the time-dependent expectation $\bar{g}(t)$, which can be approximated by an ensemble average. We denote the birth death chain related to $f_k$ by $(Y_t : t \geq 0)$ and let $Y_t^i$ be different realisations with $i = 1, \ldots, m$ in an ensemble of size $m$. With master equation (4.3), we then use the approximation

$$\bar{g}(t) \approx \langle g \rangle_m = \frac{1}{m} \sum_{i=1}^{m} g(Y_t^i) \ . \tag{4.27}$$

The dynamics of $m$ birth death chains $Y = (Y^i : i = 1, ..., m)$ related to $f_k(t)$ is defined by the generator

$$\mathcal{L}H(Y) = \sum_{i=1}^{m} g(Y^i)[H(Y - e^i) - H(Y)] + \bar{g}(t)[H(Y + e^i) - H(Y)],$$

where $e^i \in \{0,1\}^m$ is the unit vector $(e^i)_j = \delta_{i,j}$. Using the special test function $H(Y) = \sum_{i=1}^m Y^i$ for the total number of particles in the birth death chains, we get

$$\mathcal{L}H(Y) = -\sum_{i=1}^{m} g(Y^i) + \sum_{i=1}^{m} \frac{1}{m} \sum_{j=1}^{m} g(Y^j) = 0.$$

This implies that $H(Y_t) = \sum_{i=1}^m Y_t^i$ is a (non-negative) martingale and in particular that $\mathbb{E}[H(Y_t)] = \mathbb{E}[H(Y_0)]$ for all $t \geq 0$. Hence, the total density in the ensemble average $\frac{1}{m}\sum_{i=1}^m Y_t^i$ is a martingale, i.e. future expectations are equal to its present value. As opposed to zero-range processes where the total number of particles is strictly conserved, this is still a fluctuating quantity. Furthermore, the ensemble has an absorbing state at $Y^i = 0$ for all $i = 1, \ldots m$, since the approximated birth rate in this state is $\langle g \rangle_m = 0$. This leads to the fact that by fluctuations all ensembles get absorbed in state 0, and one can show that the average time to absorption scales

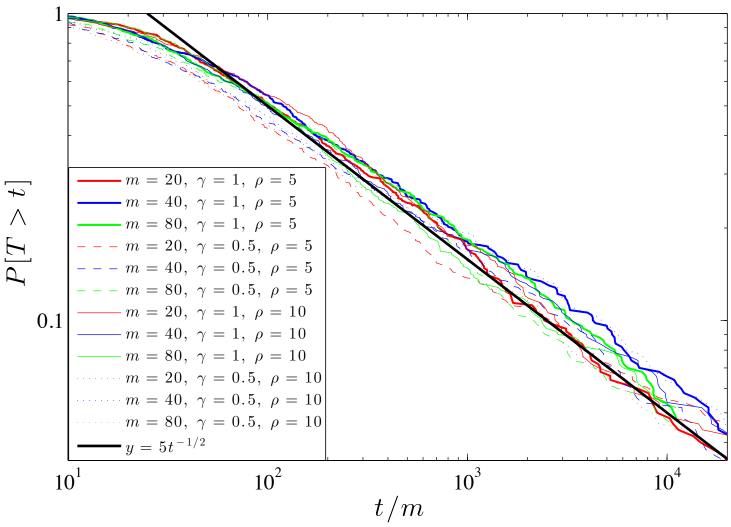



Figure 4.4: The tail distribution of the absorption time $T$ of $X_t$ for $b = 4$ against rescaled time $t/m$ confirms the scaling of $T$ being linear in $m$. The actual distribution of $T$ has the same tail proportional to $t^{-1/2}$ as simple random walks or Brownian motion.

like $m$. By considering, $H(Y)^2 = (\sum_{i=1}^m Y^i)^2$ we get

$$\begin{aligned}
\mathcal{L}(H^2) &= \sum_{i=1}^m g(Y^i)[(H-1)^2 - H^2] + \bar{g}(t)[(H+1)^2 - H^2] \\
&= \sum_{i=1}^m g(Y^i)[-2H+1] + \sum_{i=1}^m \bar{g}(t)[2H+1] \\
&= -2m\bar{g}(t)\sum_{i=1}^m Y^i + m\bar{g}(t) + 2m\bar{g}(t)\sum_{i=1}^m Y^i + m\bar{g}(t) \\
&= 2m\bar{g}(t).
\end{aligned}$$

Therefore, the quadratic variation is $[H]_t = \int_0^t \mathcal{L}H^2(Y_s)ds = 2m\int_0^t \bar{g}(s)ds$, which characterises the fluctuations of the martingale $H(Y_t)$ as discussed in Section 2.1 (see e.g. Chapter 2 in [32] for more details). This is linear in $t$ to leading order as long as $\bar{g}(t) > 0$ is bounded. With $H(Y_0) = \rho m$, the process can get absorbed in 0 when the variance reaches a level of order $(\rho m)^2$. Since $\bar{g}(s)$ converges to $\phi(\rho)$, this implies $2mT\phi(\rho) = (\rho m)^2$ to characterise the expected absorption time $T$ for large $m$, which is then approximately $m\frac{\rho^2}{2\phi(\rho)} \approx m$ as is confirmed in Figure 4.4. This implies that the coarsening process can only be truthfully represented by the ensemble in a finite time window, which is similar to the restriction in the zero-range process due to the approach to stationarity. This represents a strong limitation for numerical analysis and is illustrated in Figure 4.10 further below. A further disadvantage is that in both

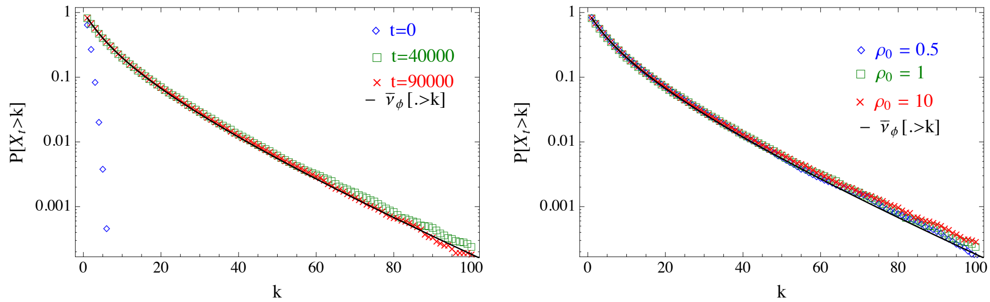



Figure 4.5: Convergence of the tail distribution for the size-biased chain $(X_t : t \geq 0)$ in the subcritical case $\rho < \rho_c$ to a size-biased version of the stationary distribution $\bar{\nu}_\phi[\cdot > k]$ (4.25) with $\phi = 0.95$ such that $R(\phi) = \rho$. The limiting distribution has an exponential tail, and the initial condition is Poisson with density $\rho_0$. Parameter values are $\gamma = 1$ with $b = 2.5$ and ensemble size $m = 10^5$. The particle density is $\rho_0 = \rho = 1 < \rho_c = 1/(b-2) = 2$ (left) while we also confirm that convergence only depends on the parameter $\rho = 1$ entering the dynamics through (4.28), and is independent of the initial density $\rho_0 = 0.5, 1$ and 10 (right).

processes, the chains related to $f_k$ and the original zero-range process, the condensed phase only covers a vanishing fraction of the ensemble or lattice, which leads to poor statistics as is illustrated in Figure 4.9 below. We can overcome these problems by simulating an ensemble of size-biased birth death chains $(X_t^i : t \geq 0)$ with master equation (4.22). The ensemble average

$$\langle g \rangle_m = \rho \sum_{i=1}^{m} g(X_t^i)/X_t^i \ , \tag{4.28}$$

has to be modified as explained in the previous section since $X_t^i$ now represents a size-biased single site process. In contrast to the $Y_t^i$ chains, the density $\rho$ explicitly enters the dynamics as a parameter. In addition, we use the ensemble to implement killing and cloning events as follows: since the killing rate of chains with occupation number $k = 1$ is equal to the total cloning rate for chains with $k > 1$, everytime we clone a chain, we kill a chain with $k = 1$ to keep the ensemble size $m$ fixed. There may be instances where no $k = 1$ chain exists in the ensemble at the time of cloning, but it turns out that for large enough ensemble sizes this happens very rarely (for our parameter values not more than 10 times in simulations up to times of order $10^6$), and we can ignore such events. They could easily be taken into account by allowing the ensemble size to grow in time, but this does not make any difference to numerical results.

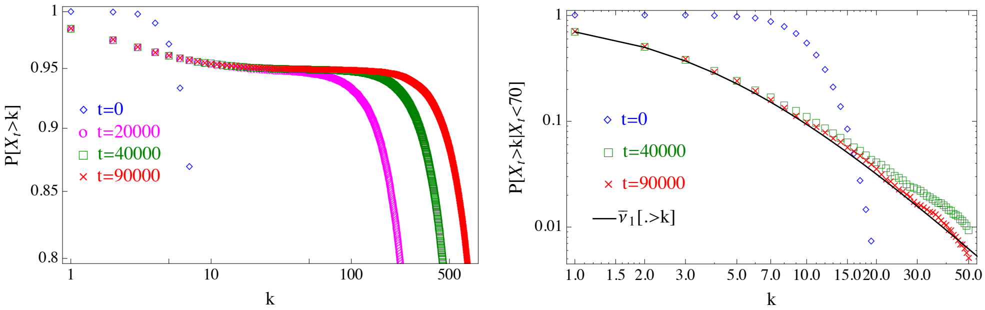



Figure 4.6: Time evolution of the tail distribution of the size-biased chain $(X_t : t \geq 0)$ in the supercritical case $\rho > \rho_c$. Parameter values are $\gamma = 1$, $b = 4$ and $\rho = 10 > \rho_c = 0.5$, the ensemble size is $m = 10^5$. A plateau emerges at the value $1 - \frac{\rho_c}{\rho} = 0.95$ characterising the phase separation as shown on the left. While the bulk part conditioned on small occupation numbers $\mathbb{P}[X_t > k \mid X_t < 70]$ converges to the tail of a size-biased stationary marginal of the zero-range process $\bar{\nu}_1$ (4.25) as shown on the right, the condensed part of the distribution keeps evolving to larger occupation numbers.

### 4.3.2 Main results

In the following we present our results comparing numerical solutions of the scaling equations with simulation results from the size-biased birth death chain and master equation (4.22) $(X_t : t \geq 0)$ with $X_t \in \mathbb{N}$. As a first test, we confirm that for subcritical densities $\rho < \rho_c$, $X_t$ eventually converges to a size-biased stationary distribution as given in the previous section. This is illustrated in Figure 4.5, where the tail distribution of the process $X_t$ converges to a size-biased version $p^\phi$ (4.25) of the stationary marginal of the zero-range process. This is independent of the actual initial condition $X_0$, the asymptotic density $\rho$ is determined by the parameter $\rho$ in (4.24) and (4.28) as shown in Figure 4.5 (right). For supercritical $\rho > \rho_c$, the distribution of $X_t$ phase separates, where with probability $\frac{\rho_c}{\rho}$ it takes small values corresponding to the bulk sites of the zero-range process. This part of the distribution again converges to $p^\phi$ with $\phi = 1$ which has now a sub-exponential tail, as is illustrated in Figure 4.6 (right). With probability $\frac{\rho - \rho_c}{\rho}$, the chain takes large values corresponding to the condensed phase. This is shown in Figure 4.6 (left) where we plot the tail $P[X_t > k]$ and see a plateau emerging at $1 - \frac{\rho_c}{\rho}$. In Figure 4.7, we compare the ensemble average for $\bar{g}$ under the ensemble of birth death chains with the spatial average in the zero-range process, and find good agreement. The fluctuations around the mean are of similar size as well since we choose $m$ and $L$ of similar size, even though much higher values of $m$ can be treated numerically without problems. Note that a single ensemble of $m = L$ chains gives the same quality of data as an average of 500 realisations of the zero-range process of size $L$. The data

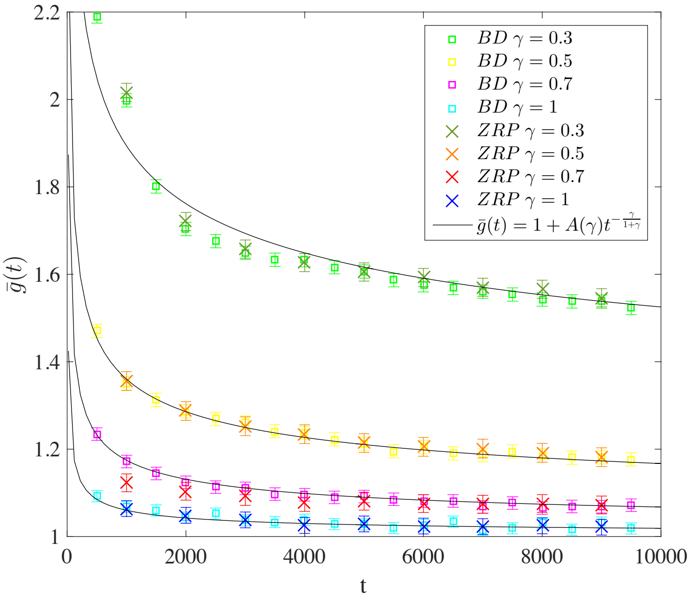
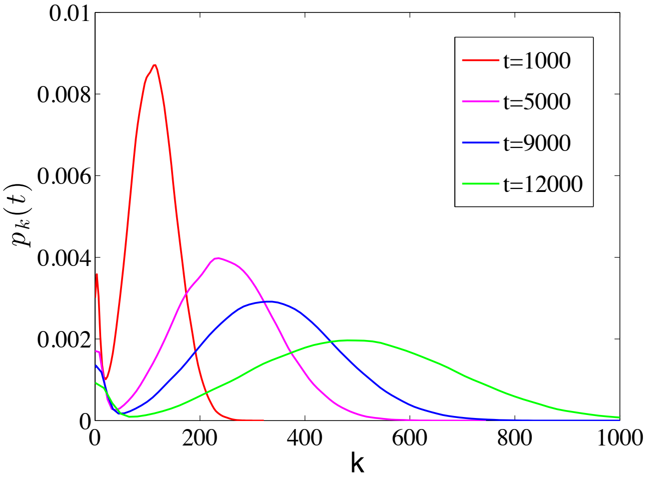



Figure 4.7: Agreement of the ensemble averages for the expected jump rate $\bar{g}$ under zero-range dynamics (ZRP) and the birth death chain $X_t$ (4.28) (BD), in comparison with the theoretical prediction $\bar{g}(t)$ in (4.15) with fitted parameters $A(\gamma)$. Parameter values are $b = 4$ and $\rho = 2$, system sizes are $L = m = 1024$. The data for ZRP have further been averaged over 500 realisations. Error bars denote standard error of mean and are comparable in both systems.

also compare well with the theoretical prediction obtained in (4.15) with a fitted constant $A$ depending on the parameter $\gamma$. By plotting the empirical distribution of $X_t$, we can see in Figure 4.8 (left) that the condensed part of the distribution has time independent mass $\frac{\rho-\rho_c}{\rho}$ and is moving to the right. Plotting against the rescaled occupation numbers $u = k\epsilon_t$ we see a data collapse confirming the predicted scaling in (4.8) as shown in Figure 4.8 (right). The rescaled distributions of the condensed part also match well with the solution of (4.17) which is shown for $\gamma = 0.5$ and $\gamma = 1$ in Figure 4.9. While for $\gamma = 1$ the theoretical prediction is indeed independent of time, there is a time dependence for $\gamma < 1$ as can be seen in the left plot. The asymptotic analysis as $t \to \infty$ is delicate in this case, see [46] for details. We also show data from a direct simulation of a zero-range process with the same numerical effort and bandwidth parameter for smoothing the density. The birth death chain obviously provides much better data for the condensed phase, and describes the distribution well, not only asymptotically but also for relatively small time values. Another interesting quantity is the second moment of the occupation numbers which



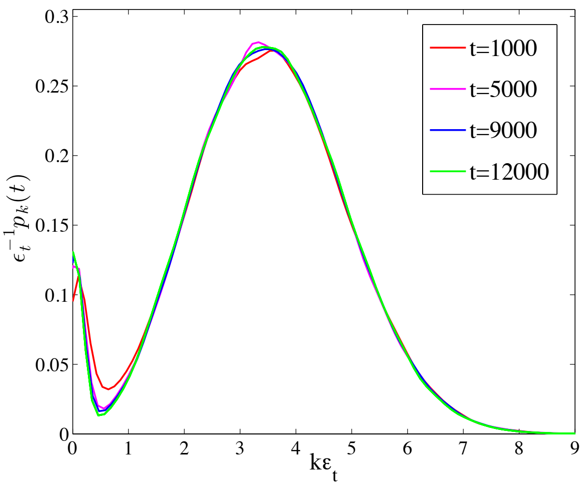
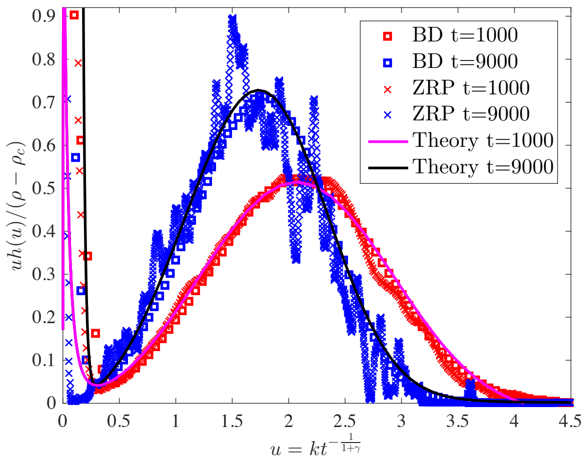




Figure 4.8: Scaling behaviour of the condensed part of the distribution $p_k(t)$, with a data collapse when plotted against the rescaled variable $u = k\epsilon_t$ as shown in the bottom. Parameter values are $\gamma = 1,\ b = 4$ and $\rho = 10$, with ensemble size $m = 10^5$.

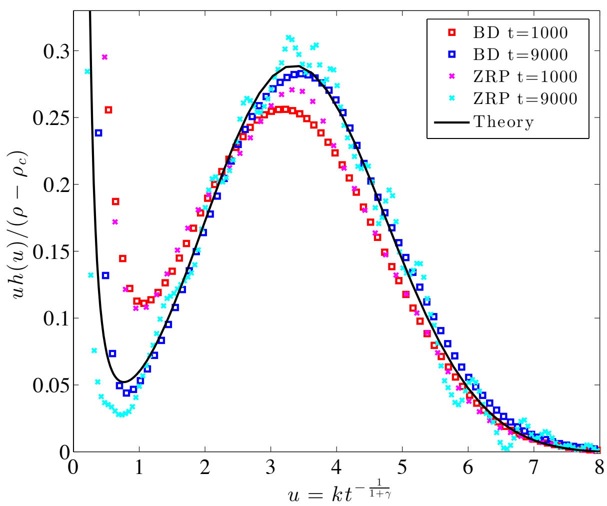
can be computed as

$$\sigma^2(t) = \rho\mathbb{E}[p_k(t)] = \rho\sum_k kp_k(t) = \sum_k k^2 f_k(t). \tag{4.29}$$

In the coarsening regime, the time evolution of $\sigma^2(t)$ is expected to follow a scaling law analogous to results in [44]. With the above expression, one can compute using




$\gamma = 0.5$




$\gamma = 1$

Figure 4.9: Normalised theoretical prediction $uh(u)$ as a solution of (4.17) and (4.18) shows good agreement with asymptotic simulation data for the birth death chain $\epsilon_t^{-1}\rho p_k(\eta)$ (BD), plotted against the rescaled variable $u = k\epsilon_t$. Data from direct zero-range simulations (ZRP) coincide but are clearly of inferior quality in particular for large times since the volume of the condensed phase decreases. Parameter values are $b = 4$, $\rho = 2$ with $\gamma = 0.5$ and 1 and ensemble size $L = m = 1024$. For ZRP, we further average over 500 realisations while for BD we only use one.

the master equation (4.3)

$$\frac{d}{dt}\sum_k k^2 f_k(t) = \sum_k g(k+1)k^2 f_{k+1}(t) + \sum_k \bar{g}(t)k^2 f_{k-1}(t) \\ - \sum_k (g(k) + \bar{g}(t))k^2 f_k(t)$$

$$
\begin{aligned}
&= \sum_{k\geq 1} g(k)(k-1)^2 f_k(t) + \sum_k \bar{g}(t)(k+1)^2 f_k(t) \\
&\quad - \sum_k (g(k)+\bar{g}(t))k^2 f_k(t) \\
&= \sum_{k\geq 1} g(k)(-2k+1) f_k(t) + \sum_k \bar{g}(t)(2k+1) f_k(t) \\
&= -2\sum_{k\geq 1}\left(1+\frac{b}{k^\gamma}\right) k f_k + \sum_k g(k) f_k(t) \\
&\quad + 2\bar{g}(t)\sum_k k f_k(t) + \bar{g}(t)\sum_k f_k(t) \\
&= -2\sum_{k\geq 1} k f_k(t) - 2b\sum_{k\geq 1} k^{1-\gamma} f_k(t) + 2\bar{g}(t)\rho + 2\bar{g}(t) \\
&= -2\rho + 2\bar{g}(t) + 2\rho\bar{g}(t) - 2b\sum_{k\geq 1} k^{1-\gamma} f_k(t).
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt}\sigma^2(t) = 2\rho(\bar{g}(t)-1) + 2\Big(\bar{g}(t) - b\sum_{k\geq 1} k^{1-\gamma} f_k(t)\Big), \tag{4.30}
$$

and for $\gamma = 1$, this simplifies further to

$$
\frac{d}{dt}\sigma^2(t) = 2\rho(\bar{g}(t)-1) + 2(\bar{g}(t) - b(1-f_0(t))).
$$

Using that $\lim_{t\to\infty} f_0(t) = \nu_1[0] = \frac{b-1}{b}$, we get for large times

$$
\frac{d}{dt}\sigma^2(t) = \Big(2\rho+2\Big)(\bar{g}(t)-1) = \Big(2\rho+2\Big)A\epsilon_t^\gamma, \tag{4.31}
$$

which can be integrated to

$$
\sigma^2(t) = Bt^{\frac{1}{1+\gamma}}, \tag{4.32}
$$

where $B = (2\rho+2)(1+\gamma)A$ is a constant related to $A$ and $\rho$. While exact for $\gamma = 1$, for $\gamma < 1$ explicit computations are not possible but numerical data strongly suggest that the second term in (4.30) does not affect the scaling of the second moment, and (4.32) remains valid when fitting constants $B$. This scaling law is plotted in Figure 4.10 which shows good agreement with the simulation data of the size-biased chains. Data from the zero-range process agree in the coarsening regime, and data from the chains related to $f_k(t)$ show large fluctuations due to the existence of the absorbing state, so are numerically not very useful.

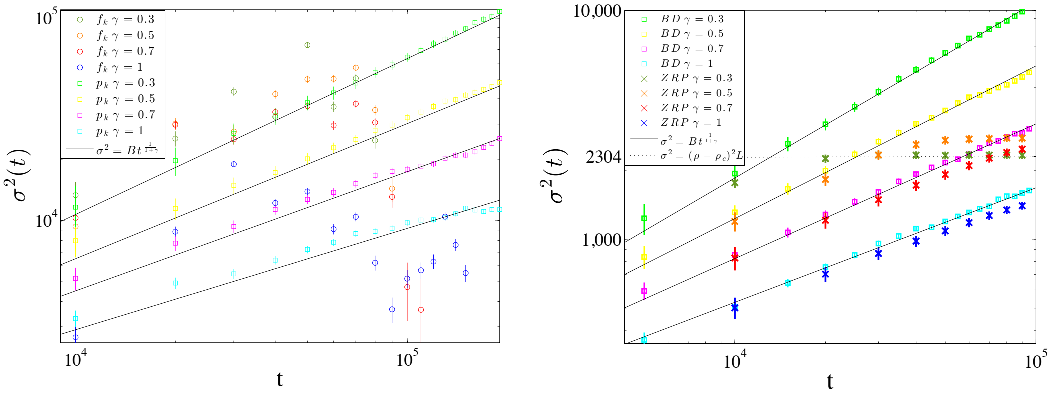



Figure 4.10: The second moment $\sigma^2(t)$ (4.29) increases asymptotically as a power law predicted in (4.32), as can be seen from simulation data of $X_t$ (shown by square boxes). We compare this to data from the birth death chains for $f_k(t)$ (left) with parameter values $b = 4$, $m = 1000$ and $\rho = 10$, and to zero-range processes (right) with $b = 4$, $\rho = 2$ and system size $L = m = 1024$. Both suffer from finite-size effects, the $f_k$ chains get eventually absorbed leading to large fluctuations at large times and the ZRP data saturate and converge to a stationary value around $(\rho - \rho_c)^2 L = 2304$, which is visible for $\gamma = 0.3$ and $0.5$ modulo finite size effects.

## 4.4 Conclusion

We apply results on the limiting single site dynamics in Chapter 3 on the zero-range process with a particular bounded jump rate of the form (2.41). We then use the mean-field master equation to study the coarsening dynamics towards condensation for the process via the site and size-biased empirical processes, which are non-linear birth death chains with master equations (4.3) and (4.22), respectively. The coarsening time scale $\epsilon_t = t^{-1/1+\gamma}$ for $\gamma \in (0, 1]$ is derived from a phase separated ansatz for the solution of the closed equation, analogously to results in [44] for $\gamma = 1$. The main novelty is the use of the size-biased birth death chain providing a strong tool to sample the coarsening dynamics without finite size effects and significantly improved statistics. This approach is generic and can be adapted to other condensing particle systems such as the inclusion process, which will be shown later in Chapter 5.

Our results have been presented for the complete graph case. In other translation invariant geometries, such as one-dimensional lattices with periodic boundary conditions, the time scales can depend on local transport properties on the lattice, in particular on the symmetry of the dynamics. Asymmetric dynamics behave in general as the complete graph independently of the dimension (cf. [44] and [55]). The scaling can be affected by the particular behaviour of first passage times and probabilities dominating the transport process of mass between cluster sites. This is slowing down coarsening for symmetric systems in one dimension with an expected scaling of $\sigma^2(t) \sim t^{\frac{1}{2+\gamma}}$ in analogy with results in [44] and [55]. In higher dimensions we expect the same scaling as for asymmetric systems and the complete graph, with logarithmic corrections in two dimensions. Mathematically this results from the mean-field approximation to derive the master equations (4.3) and (4.22) not being justified in certain symmetric systems, and it has to be adapted. In particular, the replacement of the spatially averaged jump rate $\langle g \rangle_\eta = \frac{1}{2}(g(\eta_{x-1}) + g(\eta_{x+1}))$ in (4.13) in 1-dimensional symmetric systems has to be corrected since the presence of a cluster at site $x$ modifies the occupation number at sites $x-1$ and $x+1$, resulting in a decrease of the birth rates in the birth death chain.

# Chapter 5

# Coarsening dynamics in inclusion processes and duality

## 5.1 Introduction

We have proved the convergence of the single site process of a misanthrope-type process with bounded rates to a Markovian non-linear birth death chain in Chapter 3. In this chapter, we extend the result to a particular version with unbounded jump rates, the inclusion process which has been introduced in Section 2.2.4. Recall the generator of the inclusion process

$$(\mathcal{L}_{\mathrm{IP}}h)(\eta) = \sum_{x,y\in\Lambda} q(x,y)\eta_x(d+\eta_y)(h(\eta^{x\to y}) - h(\eta)). \qquad \text{(2.43 revisited)}$$

Even though the inclusion process has rate $c(k,l) = k(d+l)$ which is unbounded and is not covered by Theorem 3.1, the single site dynamics can be derived differently. For a process with a finite number of particles on finite state space, the single site process can be written in a form of a (linear or standard) birth death chain with birth rate $\beta_k \geq 0$ and death rate $\mu_k \geq 0$ on $E = \{0, 1, ..., N\}$ whereas in the thermodynamic limit the state space is $E = \mathbb{N}_0$. Both cases will be proved in Theorem 5.1. Since the chain is linear, we can simply use the usual method of generating functions for birth death chain [63, 64, 52] to study its mean and covariance. From the mean-field master equation, we also analyse the scaling limit of the coarsening dynamics in the condensing case for $d = 0$, using the method studied for zero-range processes in Chapter 4. We also recover the size-biased process $p_k$ analogously to one we have for the zero-range process. Another approach to study inclusion processes is self-duality which allows us to study condensation also for $d = d_L \to 0$ as mention in Section

2.2.4
This chapter is organised as follows. In Section 5.2, we state the main result of convergence with proof, followed by analysis sections for both processes $f_k$ and $p_k$. In Section 5.5, we compute the covariances of the inclusion process on a complete graph using a dual process and recover some results of Section 5.3 and more general results on finite system size.

## 5.2 Coarsening dynamics in Inclusion processes

Consider the single site process $(\eta_x(t) : t \geq 0)$ for a fixed $x \in \Lambda$ (with $L = |\Lambda|$ large enough) on a complete graph with $q(x, y) = 1/(L-1)$ for all $x \neq y$ in generator (2.43).

**Theorem 5.1.** *For each $L, N \geq 1$, the process $\eta_x(t)$ is a birth death chain on the state space $E = \{0, 1, ..., N\}$ with*

$$\beta_k = \frac{N-k}{L-1}(d+k) \ \ \textit{and}\ \mu_k = k\frac{(d(L-1)+N-k)}{L-1}. \tag{5.1}$$

*In the thermodynamic limit, i.e. $L, N \to \infty$ with $\frac{N}{L} \to \rho$, $\eta_x(t)$ converges (weakly on path space) to a birth death chain on $\mathbb{N}_0$ with rates*

$$\beta_k = \rho(d+k) \ \ \textit{and}\ \mu_k = k(d+\rho), \tag{5.2}$$

*with the corresponding master equation*

$$\frac{d}{dt}f_k(t) = (k+1)(d+\rho)f_{k+1}(t) + \rho(d+(k-1))f_{k-1}(t) - (dk+2\rho k+\rho d)f_k(t), \tag{5.3}$$

*valid for all $k \geq 0$ with the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$.*

*Proof.* Consider a function $h : \mathbb{N}_0 \to \mathbb{R}$. For $\eta_x > 0$, we have

$$\begin{aligned}(\mathcal{L}_{\mathrm{IP}}h)(\eta_x) &= \frac{1}{L-1}\sum_{y\neq x}\eta_x(d+\eta_y)(h(\eta_x-1)-h(\eta_x)) \\ &\quad + \frac{1}{L-1}\sum_{y\neq x}\eta_y(d+\eta_x)(h(\eta_x+1)-h(\eta_x)) \\ &= \frac{1}{L-1}\eta_x(d(L-1)+N-\eta_x)(h(\eta_x-1)-h(\eta_x)) \\ &\quad + \frac{1}{L-1}(N-\eta_x)(d+\eta_x)(h(\eta_x+1)-h(\eta_x)),\end{aligned} \tag{5.4}$$

whereas for $\eta_x = 0$,

$$(\mathcal{L}_{\mathrm{IP}}h)(0) = \frac{1}{L-1}\sum_{y\neq x} d\eta_y(h(1)-h(0)) = \frac{N}{L-1}d(h(1)-h(0)). \tag{5.5}$$

These expressions are closed in $\eta_x$ and take the form of a generator of a birth death chain with rates $\beta_k$ and $\mu_k$ as given in (5.1). Thus, for the inclusion process, the single site dynamics $\eta_x(t)$ is a Markov process for every fixed system size. As $L, N \to \infty$ , $N/L \to \rho$,

$$\beta_k \to \rho(d+k) \;\text{ and }\; \mu_k \to k(d+\rho).$$

As a simple application of the Trotter Kurtz Theorem (see e.g. Theorem I.2.12 in [72]), the single site dynamics $\eta_x(t)$ on $E = \{0, 1, ..., N\}$ with generator (5.4), (5.5) converge to a birth death process with state space $\mathbb{N}_0$ and the stated rates. □

Note that the birth death chains are without non-linearity in the master equation unlike what we have seen before for general misanthrope processes with bounded rates. The dependence on $f_k(t)$ is only via the first moment $\rho = \sum_k kf_k(t)$, which is constant.

By spatial homogeneity, Theorem 5.1 also implies that for all $k \geq 0$ and $t \geq 0$,

$$f_k^L(t) = \mathbb{E}[F_k(\eta(t))] = \frac{1}{L}\sum_{x\in\Lambda}\mathbb{P}[\eta_x(t)=k] \to f_k(t) \tag{5.6}$$

as $L, N \to \infty, N/L \to \rho$ which is a solution of (5.3).

Therefore, for the size-biased version (4.1) we get convergence of $p_k^L(t)$ i.e.

$$p_k^L(t) = \mathbb{E}\left[\frac{kL}{N}F_k(\eta(t))\right] \to p_k(t) := \frac{k}{\rho}f_k(t), \tag{5.7}$$

for all $k \geq 1$ and $t \geq 0$.

## 5.3 Analysis of the single site limit dynamics

We denote the limit process with the master equation (5.3) by $(Y_t : t \geq 0)$ which has the generator in the form

$$\mathcal{L}_{\mathrm{BD}}h(k) = k(d+\rho)(h(k-1)-h(k)) + \rho(d+k)(h(k+1)-h(k)). \tag{5.8}$$

For $h(k) = k$,

$$\mathcal{L}_{\mathrm{BD}}k = -k(d+\rho) + \rho(d+k) = d(\rho - k). \tag{5.9}$$

This means that the process is a martingale if and only if $d = 0$. Also, for all values of $d \geq 0$, we have

$$\mathbb{E}[Y_0] = \rho \ \text{ and } \ \frac{d}{dt}\mathbb{E}[Y_t] = \mathbb{E}[\rho - Y_t].$$

This leads to $\mathbb{E}[Y_t] = \rho$ for all $t \geq 0$, corresponding to conservation of mass of the original dynamics.
For $h(k) = k^2$, we have

$$\mathcal{L}_{\mathrm{BD}} k^2 = k(d+\rho)(1-2k) + \rho(d+k)(1+2k) = -2dk^2 + (2\rho + 2\rho d + d)k.$$

Taking expectation, we get

$$\frac{d\sigma^2(t)}{dt} = \frac{d}{dt}\mathbb{E}[Y_t^2] = 2d\sigma^2(t) + 2\rho(\rho + \rho d + d),$$

which has solution

$$\sigma^2(t) = \sigma^2(0) + \frac{\rho^2}{d}(1 - e^{-2dt}), \quad \text{for } d > 0. \tag{5.10}$$

This is consistent with the simpler case $d = 0$, where we get

$$\sigma^2(t) = \sigma^2(0) + 2\rho^2 t. \tag{5.11}$$

For example, if particles are distributed uniformly at time 0, $\eta_x(0)$ is binomial with a Poissonian limit Poi$(\rho)$ analogously to (3.31). In that case $\sigma^2(0) = \rho(\rho+1)$. When $d > 0$, to find the stationary distribution $f_k^\rho$ for the birth death chain, we can use detailed balance (2.16), i.e. $f_k^\rho \beta_k = f_{k+1}^\rho \mu_{k+1}$, to get

$$f_k^\rho = \frac{\beta_0\beta_1\beta_2\cdots\beta_{k-2}\beta_{k-1}}{\mu_1\mu_2\cdots\mu_k} f_0^\rho = \frac{\rho^k}{(\rho+d)^k}\frac{\Gamma(d+k)}{k!\Gamma(d)} f_0^\rho.$$

As expected, this is identical to the single site marginal of the stationary product measure of the inclusion process (2.30) with weight (2.44). From this we know the normalization and have

$$f_k^\rho = \frac{\Gamma(d+k)}{k!\Gamma(d)}\left(\frac{d}{\rho+d}\right)^d\left(\frac{\rho}{\rho+d}\right)^k, \quad \text{for all } k \geq 0, \tag{5.12}$$

where the fugacity parameter in the original formulation in Section 5.5 is given by $\phi(\rho) = \frac{\rho}{\rho+d}$. Note that conservation of $\rho = \mathbb{E}[Y_t]$ leads to a family of stationary distributions for each $\rho > 0$, analogously to ZRP results in Chapter 4. For $d = 0$, the stationary distribution degenerates to $f_k^\rho = \delta_{k,0}$ for all $\rho \geq 0$, and an absorbing state in 0 drives condensation in the original inclusion process where all particles accumulate on one site. Formally, this corresponds to the critical density $\rho_c = 0$ as we discussed in Section 2.2.4.

### 5.3.1 Generating function for $d = 0$

Generating functions are widely used and play an important role in probability as they can uniquely characterise the probability distribution of a process [82, 52]. For $d = 0$, the master equation (5.3) becomes

$$\frac{d}{dt} f_k(t) = \rho(k+1) f_{k+1}(t) + \rho(k-1) f_{k-1}(t) - 2\rho k f_k(t), \tag{5.13}$$

for $k \in \mathbb{N}_0$ with the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$. Now $(Y_t : t \geq 0)$ is a standard birth death process with the same birth and death rates $\beta_k = \mu_k = \rho k$. Note that the initial condition $Y_0$ is a random variable with $\mathbb{E}[Y_0] = \rho$. We also have $\beta_0 = \mu_0 = 0$, so we have the absorbing state at $k = 0$. The process is simply a time change of a continuous time simple symmetric random walk with absorption at 0. Consider the generating function

$$G(z,t) = \sum_{k=0}^{\infty} z^k f_k(t) = \mathbb{E}[z^{Y_t} \mid Y_0],$$

conditional on the initial state $Y_0$.
Multiply the $k$-th equation of (5.3) by $z^k$ and sum to obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} z^k \frac{d}{dt} f_k(t) &= \rho \sum_{k=0}^{\infty} (k+1) z^k f_{k+1}(t) + \rho \sum_{k=1}^{\infty} (k-1) z^k f_{k-1}(t) - 2\rho \sum_{k=0}^{\infty} k z^k f_k(t) \\
&= \rho \sum_{k=0}^{\infty} k z^{k-1} f_k(t) + \rho z^2 \sum_{k=0}^{\infty} k z^{k-1} f_k(t) - 2\rho z \sum_{k=0}^{\infty} k z^{k-1} f_k(t).
\end{aligned}$$

Hence, we have

$$\frac{\partial G}{\partial t} = \rho \frac{\partial G}{\partial z} + \rho z^2 \frac{\partial G}{\partial z} - 2\rho z \frac{\partial G}{\partial z} = \rho(z-1)^2 \frac{\partial G}{\partial z}, \tag{5.14}$$

with boundary conditions $G(z,0) = z^{Y_0}$ for $z \in [0,1]$ and $G(1,t) = 1$, $t \geq 0$.
The solution can be found using the method of characteristics and is constant along curves $t \mapsto z_t$, satisfying

$$\frac{dz_t}{dt} = \rho(z_t - 1)^2.$$

Considering $z_t = z$, it follows that

$$-\frac{1}{\rho(z-1)} + \frac{1}{\rho(z_0-1)} = -t, \quad \text{or} \quad z_0 = \frac{\rho t(1-z)+z}{\rho t(1-z)+1}.$$

With the uniqueness of solution of (5.14) and $z_0 = Y_0$, we have

$$G(z,t) = \left(\frac{\rho t(1-z)+z}{\rho t(1-z)+1}\right)^{Y_0}. \tag{5.15}$$

For given initial condition $Y_0 \in \mathbb{N}_0$, the probability of being absorbed in 0 by time $t$ is $G(0,t) = \left(\frac{\rho t}{\rho t+1}\right)^{Y_0}$. Taking $\lim_{t\to\infty} G(0,t) = 1$, we know that absorption is certain eventually for all $Y_0 \in \mathbb{N}_0$. The absorption probability can be written as

$$f_0(t) = G(0,t) = \sum_{k\geq 0} \left(\frac{\rho t}{\rho t+1}\right)^{Y_0} f_k(0), \tag{5.16}$$

with expectation over the initial distribution $f(0)$ which has first moment $\rho$. For example, for $\rho = 1$ and deterministic $Y_0 = 1$, this is simply $f_0(t) = t/(t+1)$.
To find the mean and variance of $Y_t$, we have

$$\mathbb{E}[Y_t \mid Y_0] = \frac{d}{dz} G(z,t) \mid_{z=1} = Y_0. \tag{5.17}$$

Since this holds for all $Y_0$ and due to the time homogeneous Markov property, $(Y_t : t \geq 0)$ is a martingale as mentioned in (5.9).
With $G''(1,t) = 2\rho Y_0 t + Y_0(Y_0-1)$, we get

$$\mathbb{E}[Y_t^2 \mid Y_0] = 2\rho Y_0 t + Y_0^2. \tag{5.18}$$

Taking expectation with respect to the initial distribution $f(0)$, we also recover (5.11). In principle, we can recover the full distribution from the generating function (5.15) via

$$f_k(t) = \frac{d^k}{dz^k} \frac{G(z,t)}{k!} \mid_{z=0}, \tag{5.19}$$

but it does not lead to useful formulas in this case.

Instead, we will analyze the scaling solution more directly in Section 5.4. Since we know

$$\mathbb{E}[Y_t \mid Y_0] = \mathbb{P}[Y_t = 0 \mid Y_0] \cdot 0 + \mathbb{P}[Y_t > 0 \mid Y_0]\, \mathbb{E}[Y_t \mid Y_0, Y_t > 0],$$

we can also compute

$$\mathbb{E}[Y_t \mid Y_0, Y_t > 0] = \frac{Y_0}{1 - f_0(t)} = \frac{Y_0}{1 - \left(\frac{\rho t}{\rho t + 1}\right)^{Y_0}} \simeq \rho t \text{ as } \ t \to \infty, \tag{5.20}$$

independently of $Y_0$. Here we use $\left(\frac{\rho t}{\rho t+1}\right)^{Y_0} \simeq \left(1 - \frac{Y_0}{\rho t+1}\right)$ as $t \to \infty$, and will revisit this in Section 5.4.

### 5.3.2 Generating function for case $d > 0$

Similarly to the case $d = 0$, we can derive an explicit formula for the generating function. We have

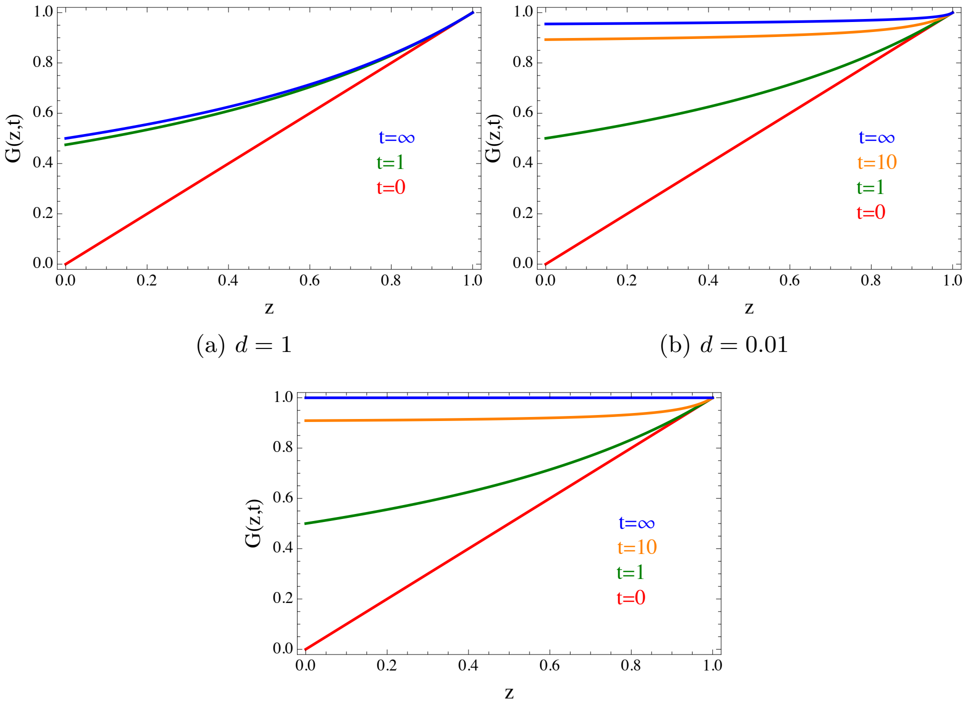
$$\begin{aligned}
\frac{\partial G}{\partial t} &= (d+\rho)\sum_{k=0}^{\infty}(k+1)z^k f_{k+1}(t) + \rho\sum_{k=0}^{\infty}(d+(k+1))z^k f_{k-1}(t) \\
&\quad - \sum_{k=0}^{\infty}(dk + 2\rho k + \rho d)z^k f_k(t) \\
&= (d+\rho)\sum_{k=0}^{\infty} kz^{k-1} f_k(t) + \rho d\sum_{k=0}^{\infty} z^k f_{k-1}(t) + \rho\sum_{k=0}^{\infty} kz^{k+1} f_k(t) \\
&\quad - (2\rho + d)\sum_{k=0}^{\infty} kz^k f_k(t) - \rho d\sum_{k=0}^{\infty} z^k f_k(t) \\
&= (d+\rho)\frac{\partial G}{\partial z} + \rho z^2\frac{\partial G}{\partial z} - (2\rho+d)z\frac{\partial G}{\partial z} + \rho dz G(z,t) - \rho d G(z,t) \\
&= (\rho z^2 - (2\rho + d)z + \rho + d)\frac{\partial G}{\partial z} + \rho d(z-1)G(z,t) \\
&= (1-z)(d+\rho-\rho z)\frac{\partial G}{\partial z} + \rho d(z-1)G(z,t).
\end{aligned}$$

Solving this with boundary condition $G(z, 0) = z^{Y_0}$ gives

$$\begin{aligned}
G(z,t) = &\frac{1}{(d+\rho(1-z))^d}\left(\frac{de^{dt}(d+\rho(1-z))}{de^{dt} + (e^{dt}-1)\rho(1-z)}\right)^d \\
&\cdot \left(\frac{(d+\rho)(z-1) + e^{dt}(d+\rho(1-z)}{\rho(z-1) + e^{dt}(d+\rho(1-z))}\right)^{Y_0}.
\end{aligned} \tag{5.21}$$

We also get

$$\mathbb{E}[Y_t \mid Y_0] = \frac{d}{dz}G(z,t) \mid_{z=1} = Y_0 e^{-dt} + (1-e^{-dt})\rho. \tag{5.22}$$




(a) $d = 1$

(b) $d = 0.01$

(c) $d = 0$

Figure 5.1: The generating function $G(z,t)$ for the case $d = 1$ (top left), $d = 0.01$ (top right) and $d = 0$ (bottom).

As mentioned earlier, $(Y_t : t \geq 0)$ is not martingale but

$$\mathbb{E}^{f(0)}[Y_t] = \rho$$

since the initial condition $f(0)$ has first moment $\rho$. With $G''(1,t)$ and computing $\mathbb{E}^{f(0)}[Y_t^2]$ we again recover (5.10). It can be seen in Figure 5.1 that the generating function is convex. For the case $d = 0$, $\lim_{t\to\infty} G(z,t) = 1$ for all $z \in [0,1]$. For any $d > 0$, the generating function has a nontrivial limit corresponding to the stationary distribution $f^\rho$ as given in (5.12).

## 5.4 Scaling limit for the dynamics with $d = 0$

It has been shown in [18, 22] that condensation in the inclusion process on regular lattices occurs when $d = d_L \to 0$ as $L \to \infty$, $N/L \to \rho > 0$ and $d_L L \to 0$. Here, considering dynamics on the complete graph, we study the case $d = 0$ for simplicity. This would lead to degenerate dynamics on regular lattices with many absorbing states, and has therefore, not been addressed in previous work.

### 5.4.1 Analysis of $F_k$

Analogously to (4.6) for ZRP, the solution to (5.13) phase separates into a bulk and condensed part

$$f_k(t) = f_k^{\mathrm{bulk}}(t) + f_k^{\mathrm{cond}}(t). \tag{5.23}$$

Since $\rho_c = 0$ and $k = 0$ is absorbing, the bulk part is very simple and given by

$$f_k^{\mathrm{bulk}}(t) = f_0(t)\delta_{k,0} \to \delta_{k,0} \quad \text{as} \quad t \to \infty, \tag{5.24}$$

using (5.16). In the following we focus on the condensed part $f_k^{\mathrm{cond}}(t) = f_k(t)\mathbb{I}_{\mathbb{N}}(k)$ for $k > 0$. Similarly to the ZRP, we assume that $f_k^{\mathrm{cond}}(t)$ takes the scaling form

$$f_k^{\mathrm{cond}}(t) = \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t. \tag{4.8 revisited}$$

This leads to

$$\begin{aligned}
\frac{\dot{\epsilon}_t}{\epsilon_t}[uh'(u) + 2h(u)] &= \rho\left(\frac{u}{\epsilon_t} + 1\right)\left[h(u) + \epsilon_t h'(u) + \frac{{\epsilon_t}^2}{2}h''(u)\right] \\
&\quad + \rho\left(\frac{u}{\epsilon_t} - 1\right)\left[h(u) - \epsilon_t h'(u) + \frac{{\epsilon_t}^2}{2}h''(u)\right] \\
&\quad - 2\rho\frac{u}{\epsilon_t}h(u) \\
&= 2\rho\epsilon_t h'(u) + \rho u\epsilon_t h''(u).
\end{aligned}$$

With $\epsilon_t = \frac{1}{\rho t}$ and thus $\frac{\dot{\epsilon}_t}{\epsilon_t^2} = -\rho$, we have

$$uh''(u) + (2 + u)h'(u) + 2h(u) = 0, \tag{5.25}$$

which has the solution $h(u) = Ce^{-u}$ for some constant $C$. Since we know

$$\rho = \int_0^\infty uh(u)du = \int_0^\infty Cue^{-u}du = C,$$

we have

$$h(u) = \rho e^{-u}. \tag{5.26}$$

Note that with (5.20)

$$\frac{1}{\epsilon_t} = \rho t = \mathbb{E}[Y_t \mid Y_t > 0] \text{ as } t \to \infty, \tag{5.27}$$

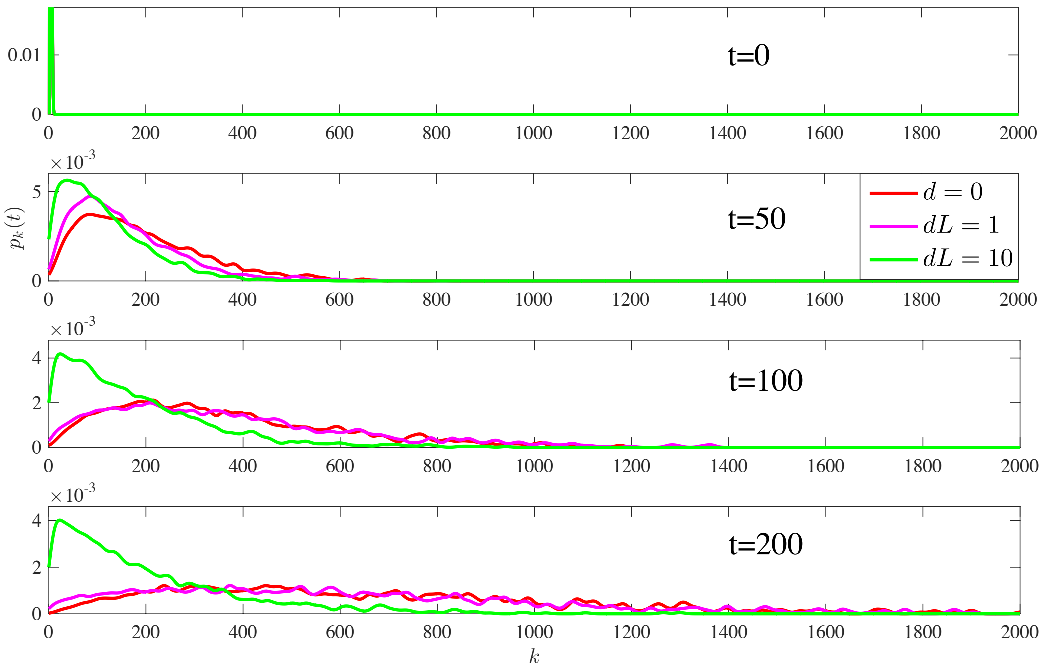



Figure 5.2: $p_k(t)$ of system size $L = 1024$ for $\rho = 2$ from the simulation of inclusion processes on complete graph with $d = 0$ (red) and $dL = 1$ (magenta) and $dL = 10$ (blue) at time 10, 50, 100 and 200 from top to bottom, respectively.

so $1/\epsilon_t$ can be interpreted as the expectation of the condensed part. We also have as $t \to \infty$,

$$\sum_{k>0} f_k^{\text{cond}}(t) = 1 - f_0(t) \simeq \frac{1}{t} \simeq \epsilon_t \int h(u)du,$$

the total probability in the condensed phase which is consistent with (5.26).

### 5.4.2 Analysis of $P_k$

Analogously to the analysis of $P_k$ for the ZRP in Section 4.2.3, using (5.3) and the relation (5.7), we get the following mean-field master equation.
For $k = 1$,

$$\begin{aligned}\frac{d}{dt}p_1(t) &= (d+\rho)p_2(t) + \rho d\frac{1}{\rho}f_0(t) - [d+\rho+\rho(d+1)]\,p_1(t)\\ &= (d+\rho)p_2(t) - 2\rho(d+1)p_1(t) + \left(\sum_{k\geq 2}[d - \frac{1}{k}d\rho]p_k(t)\right).\end{aligned}$$

For $k \geq 2$,

$$\begin{aligned}\frac{d}{dt}p_k(t) = k(d+\rho)p_{k+1}(t) + \frac{k}{k-1}\rho(d+(k-1))p_{k-1}(t) \\ - \left[(k-1)(d+\rho) + \frac{k+1}{k}\rho(d+k)\right]p_k(t) \\ + \left[\frac{1}{k}d\rho - d\right]p_k(t).\end{aligned}$$

We can check that total probability is conserved i.e.

$$\frac{d}{dt}\sum_{k\geq 1} p_k(t) = \sum_{k\geq 2}\left[d + \frac{1}{k}d\rho\right]p(t) + \sum_{k\geq 2}\left[\frac{1}{k}d\rho - d\right]p_k(t) = 0.$$

Therefore, we again have a birth death chain with killing/cloning with rates

$$\begin{aligned} &\text{birth rate} && \frac{k+1}{k}\rho(d+k), \quad \text{for } k > 0\ , \\ &\text{death rate} && (k-1)(d+\rho), \quad \text{for } k > 1\ , \\ &\text{rate from } k \text{ to } 1 && \left(d - \frac{1}{k}d\rho\right)_+, \quad \text{for } k > 1\ , \\ &\text{cloning rate} && \left(\frac{1}{k}d\rho - d\right)_+, \quad \text{for } k > 1\ , \\ &\text{killing rate} && \sum_{k>1}\left(\frac{1}{k}d\rho - d\right)_+, \quad \text{for } k = 1\ , \end{aligned} \tag{5.28}$$

where we again denote by $(\cdot)_+ = \max\{0, (\cdot)\}$ the positive part of the expression. It can be seen from Figure 5.2 that when $d$ is small enough e.g. $d \lesssim 1/L$, $p_k(t)$ almost coincide with values for $d = 0$. Hence, we can study the case $d = 0$ to understand the coarsening dynamics.

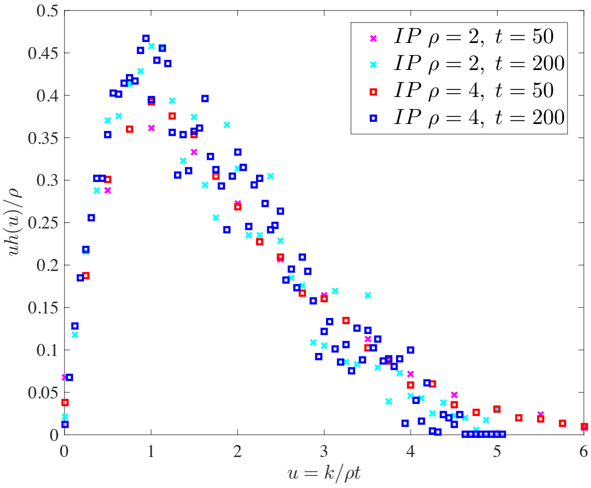
For the case $d = 0$ in $p_k(t)$, we have the size-biased birth death chain $(X_t : t \geq 0)$ on $E = \mathbb{N}$ with birth and death rates

$$\beta_k = \rho(k+1) \text{ and } \mu_k = \rho(k-1), \tag{5.29}$$

corresponding to the master equation

$$\frac{d}{dt}p_k(t) = \rho k p_{k+1}(t) + \rho k p_{k-1}(t) - 2\rho k p_k(t), \tag{5.30}$$

for all $k \geq 1$ with the convention $p_0(t) = 0$.
This is different to the process $(Y_t : t \geq 0)$ of the $f_k$ as it is not symmetric with small



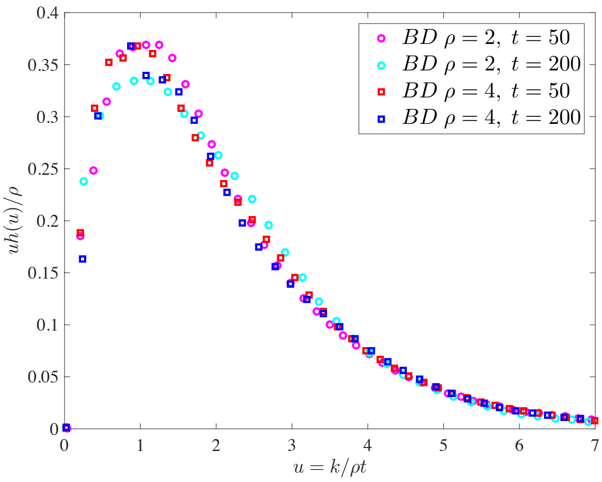




Figure 5.3: Scaling behaviour of the condensed part of the distribution $p_k(t)$ of the inclusion process on complete graph (left) and the birth death process (5.30) (right), showing a data collapse when plotted against the rescaled variable $u = \frac{k}{\rho t}$. Parameter values are $d = 0, \rho = 2, 4$ and ensemble size $L = m = 1024$.

drift to the right and also has no absorbing at state 0. Since there are no cloning and killing rates, the chain $X_t$ can be simulated as a usual birth death chain. In Figure 5.3, we plot the distribution $p_k$ against the rescaled occupation number $u = k\epsilon_t$. Note that a single birth death chain gives same quality of data as an average of 500 realisations of the inclusion process of the same size. As before, the size-biased chain $(X_t : t \geq 0)$ provides a powerful tool (due to lack of absorption) to sample from the condensed part of the distribution as seen in Figure 5.3. We again check the second moment of the process using the $p_k$ chain. We multiply the $k$-th equation of (5.30)

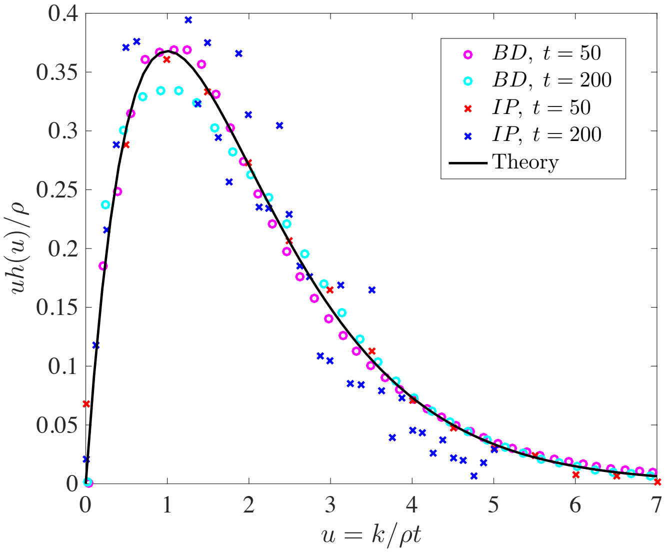



Figure 5.4: Normalised theoretical prediction $uh(u)$ as solution of (5.26), plotted against the rescaled variable $u = \frac{1}{\rho t}$. Data from direct IP simulations with averaging over 500 realisations coincide but are clearly of low quality comparing to the size-biased birth death process method which we only use one realisation of $m = L$ copies. Parameter values are $d = 0$, $\rho = 2$ and ensemble size $L = m = 1024$, the solution of (5.26) is given by a full line.

by $k$ and sum to obtain $\frac{d}{dt} \sum_k kp_k(t) = 2\rho$. This leads to

$$\sum_k kp_k(t) = 2\rho t + C \quad \text{with} \quad C = \frac{\mathbb{E}[Y_0^2]}{\rho}.$$

Hence, we recover again the previous equation (5.11),

$$\sigma^2(t) = \mathbb{E}[f_k] = \rho \mathbb{E}[p_k] = 2\rho^2 t + \sigma^2(0). \tag{5.31}$$

## 5.5 Analysis using duality

As discussed in Section 2.2.4, a condensation occurs also for positive $d = d_L \to 0$. This is not accessible with previous methods based on the limiting single site dynamics. In this section, we compute variance and covariance of the process in order to understand how coarsening occurs starting from a homogeneous product measure using a dual process containing only two particles, which has been previously studied for the nearest neighbour symmetry inclusion process in [17].

### 5.5.1 Time dependent variances

In the following, we derive a formula for the second moment of the occupation numbers for fixed system size, i.e.,

$$C_{xx}^L(t) := \mathbb{E}^{\nu_\rho}[\eta_x^2(t)]. \tag{5.32}$$

In our complete graph dynamics with a fixed number of particle $N$, we can easily derive the (non-normalised) covariance as

$$C_{xy}^L(t) := \mathbb{E}[\eta_x(t)\eta_y(t)] = \frac{N^2 - LC_{xx}^L(t)}{L(L-1)}, \tag{5.33}$$

since we know

$$\sum_{x<y} \eta_x\eta_y = \frac{1}{2}\left(\sum_{x,y} \eta_x\eta_y - \sum_x \eta_x^2\right),$$

and therefore

$$\frac{L(L-1)}{2}\mathbb{E}[\eta_x(t)\eta_y(t)] = \frac{1}{2}(N^2 - LC_{xx}^L(t)).$$

**Proposition 5.1.** *For $x \neq y \in \Lambda$, and for every initial product measure $\nu_\rho$ with density $\rho$ and second moment $\sigma^2(0)$ we have*

$$C_{xx}^L(t) = \sigma^2(0)\mathbb{P}^{x,x}[X_t = Y_t] + \left(\frac{d\rho(1+\rho)+\rho^2}{d}\right)\mathbb{P}^{x,x}[X_t \neq Y_t]. \tag{5.34}$$

*Here $X_t$ and $Y_t$ denote the particle positions for an inclusion process with two particles on the lattice $\Lambda$, and $\mathbb{P}^{x,y}$ the path measure with initial values $X_0 = x, Y_0 = y$.*

*Proof.* Let $(\eta(t) : t \geq 0)$ be a general inclusion process $(\eta(t) : t \geq 0)$ starting from $\eta$ whereas $(\xi(t) : t \geq 0)$ is another inclusion process with two particles with initial position $x, y \in \Lambda$. Denote the particle positions at time $t$ by $X_t$ and $Y_t$ and write $\xi(t) = \delta_{X_t} + \delta_{Y_t}$. Consider the initial position of the two-particle inclusion process to be on the same site i.e. $x = y$. We can write

$$\mathbb{E}^\eta[D(2\delta_x, \eta(t))] = \mathbb{E}^{x,x}[D(\delta_{X_t} + \delta_{Y_t}, \eta)],$$

as (2.47) also holds for these two processes. We know that the duality function (2.46) can be simply written as

$$D(2\delta_x, \eta(t)) = \frac{\eta_x(t)(\eta_x(t)-1)}{d(d+1)}.$$

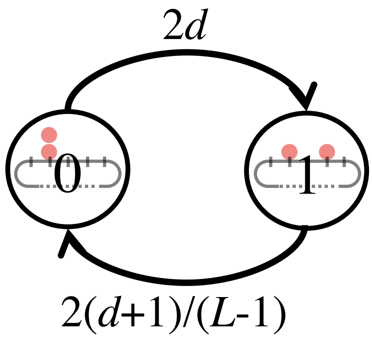



Figure 5.5: Two dual particle system of the inclusion process.

Then, taking expectation and using the duality relation, we get

$$\begin{aligned}
&\mathbb{E}^{\eta}[\eta_x(t)(\eta_x(t)-1)] \\
&= d(d+1)\mathbb{E}^{\eta}[D(2\delta_x, \eta(t))] \\
&= d(d+1)\mathbb{E}^{x,x}[D(\delta_{X_t}+\delta_{Y_t}, \eta)] \\
&= d(d+1)\left(\mathbb{E}^{x,x}\left[\frac{\eta_{X_t}(\eta_{X_t}-1)}{d(d+1)}\mathbb{I}(X_t=Y_t)\right] + \mathbb{E}^{x,x}\left[\frac{\eta_{X_t}}{d}\frac{\eta_{Y_t}}{d}\mathbb{I}(X_t\neq Y_t)\right]\right),
\end{aligned}$$

for all $\eta$.
If we take the initial distribution to be translation invariant with first moment $\rho$ and second moment $\sigma^2(0)$, then we have

$$\mathbb{E}^{\nu_\rho}[\eta_x(t)^2] - \rho = (\sigma^2(0)-\rho)\mathbb{P}^{x,x}(X_t=Y_t) + \frac{d+1}{d}\rho^2\mathbb{P}^{x,x}(X_t\neq Y_t), \tag{5.35}$$

which gives (5.34) as required. □

### 5.5.2 Exact computations for two dual particles

Since we know that the inclusion process is self dual, we consider the process with only two particles at positions $X_t$ and $Y_t$. In the complete graph case, the process $Z_t = \mid Y_t - X_t \mid$ takes values on the state space $\{0,1\}$ corresponding to both particles being on the same site, i.e. $Z_t = 0$, or on two different sites, i.e. $Z_t = 1$. Two particles at different location jump onto the same site with rate $\frac{1}{L-1}\cdot 1\cdot(d+1)$ and there are 2 possibilities, so the total rate is $\frac{2(d+1)}{L-1}$. On the other hand, the particles on the same site jump apart with rate $\frac{1}{L-1}\cdot 2\cdot(d+0)$ and there are $L-1$ other sites to jump to, so the total rate is $2d$. This leads to the Q-matrix for $(Z_t : t\geq 0)$ on $\{0,1\}$

$$Q = \begin{pmatrix} -2d & 2d \\ \frac{2(d+1)}{L-1} & -\frac{2(d+1)}{L-1} \end{pmatrix}.$$

We can diagonalise $Q$ which has eigenvalues 0 and $-\frac{2(1+dL)}{L-1}$ to obtain $Q = U\Lambda U^{-1}$ where

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2(1+dL)}{L-1} \end{pmatrix}, \text{ and } U = \begin{pmatrix} 1 & 2d(L-1) \\ 1 & -2(d+1) \end{pmatrix}.$$

Therefore,

$$\begin{aligned} P_t = e^{tQ} &= U \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{2(1+dL)}{L-1}t} \end{pmatrix} U^{-1} \\ &= \frac{1}{(1+dL)} \begin{pmatrix} (d+1) + d(L-1)e^{-\frac{2(1+dL)}{L-1}t} & d(L-1)[1-e^{-\frac{2(1+dL)}{L-1}t}] \\ (d+1)[1-e^{-\frac{2(1+dL)}{L-1}t}] & d(L-1) + 2(d+1)e^{-\frac{2(1+dL)}{L-1}t} \end{pmatrix}. \end{aligned}$$

As $t \to \infty$, we have

$$P_t \to \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix},$$

where $\alpha = \frac{d+1}{dL+1}$. Hence,

$$\mathbb{P}(Z_t = i) \to \begin{cases} \alpha & \text{if } i = 0, \\ 1-\alpha & \text{if } i = 1, \end{cases} \tag{5.36}$$

for any initial distribution of $Z_0$ and $\pi = (\alpha, 1-\alpha)$ is the stationary distribution. In our case with $Z_0 = 0$,

$$C^L_{xx}(t) = \sigma^2(0)\mathbb{P}^0[Z_t = 0] + \left(\frac{d\rho(1+\rho)+\rho^2}{d}\right)\mathbb{P}^0[Z_t = 1]. \tag{5.37}$$

Since we know

$$\mathbb{P}^0[Z_t = 1] = 1 - \mathbb{P}^0[Z_t = 0],$$

and

$$\mathbb{P}^0[Z_t = 0] = (P_t)_{00} = \frac{1}{1+dL}[(d+1) + d(L-1)e^{-\frac{2(1+dL)}{L-1}t}],$$

we get

$$C^L_{xx}(t) = \sigma^2(0) + \frac{\rho^2(L-1)}{1+dL}(1-e^{-\frac{2(1+dL)}{L-1}t}). \tag{5.38}$$

It can be seen that for $L \to \infty$,

$$C^L_{xx}(t) \to \sigma^2(t),$$

is consistent with (5.10) for fixed $d > 0$ as $L \to \infty$. If $d = 0$ or $d = d_L \to 0$ as $L \to \infty$, $N/L \to \rho > 0$ and $d_L L \to 0$, it is consistent with (5.11), so (5.38) provides the most general result.

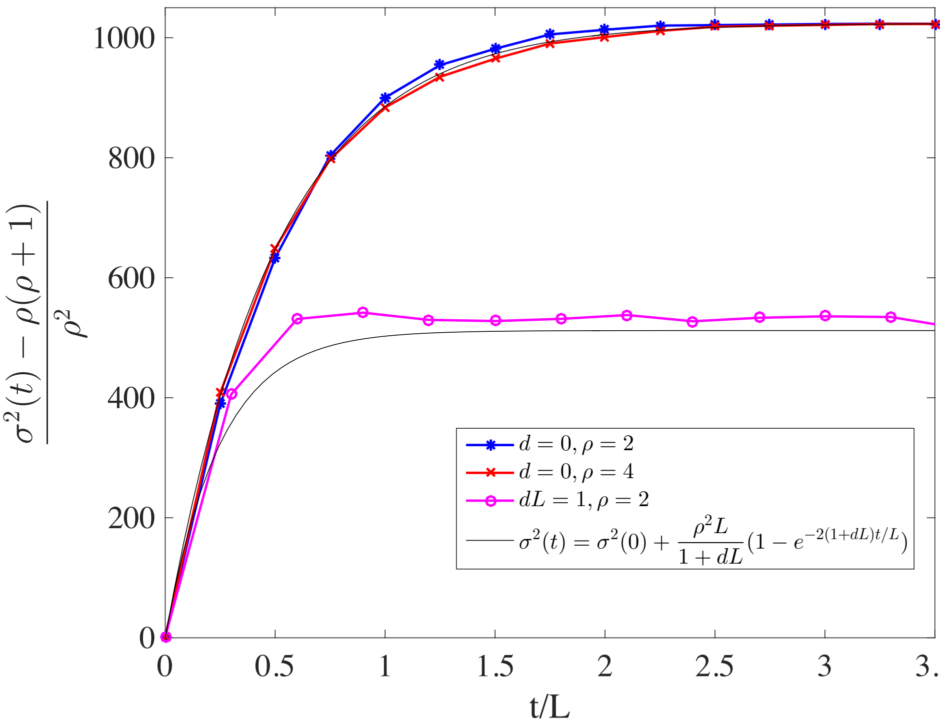



Figure 5.6: The second moment $\sigma^2(t)$ for Poi($\rho$) initial data increases as predicted in (5.38), as can be seen from simulation data for $L = 1024$, averaged over 500 realisations.

## 5.6 Observing multiple sites

Similarly to Theorem 5.1, we can derive explicit Markovian dynamics for observing multiple sites in the inclusion process. If we consider two lattice sites instead of one, we can define a generator for this process as follow,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{IP}} h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} &= \frac{1}{L-1} \sum_{z \neq x \neq y} \eta_z (d+\eta_x) \left[ h \begin{pmatrix} \eta_x + 1 \\ \eta_y \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \eta_y (d+\eta_x) \left[ h \begin{pmatrix} \eta_x + 1 \\ \eta_y - 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \sum_{z \neq x \neq y} \eta_x (d+\eta_z) \left[ h \begin{pmatrix} \eta_x - 1 \\ \eta_y \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \eta_x (d+\eta_y) \left[ h \begin{pmatrix} \eta_x - 1 \\ \eta_y + 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \sum_{z \neq x \neq y} \eta_z (d+\eta_y) \left[ h \begin{pmatrix} \eta_x \\ \eta_y + 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \sum_{z \neq x \neq y} \eta_y (d+\eta_z) \left[ h \begin{pmatrix} \eta_x \\ \eta_y - 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&= \frac{1}{L-1} (N-\eta_x-\eta_y)(d+\eta_x) \left[ h \begin{pmatrix} \eta_x + 1 \\ \eta_y \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \eta_y (d+\eta_x) \left[ h \begin{pmatrix} \eta_x + 1 \\ \eta_y - 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \eta_x \left( d + \frac{(N-\eta_x-\eta_y)}{L-1} \right) \left[ h \begin{pmatrix} \eta_x - 1 \\ \eta_y \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} \eta_x (d+\eta_y) \left[ h \begin{pmatrix} \eta_x - 1 \\ \eta_y + 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \frac{1}{L-1} (N-\eta_x-\eta_y)(d+\eta_y) \left[ h \begin{pmatrix} \eta_x \\ \eta_y + 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right] \\
&+ \eta_y \left( d + \frac{(N-\eta_x-\eta_y)}{L-1} \right) \left[ h \begin{pmatrix} \eta_x \\ \eta_y - 1 \end{pmatrix} - h \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} \right].
\end{aligned}
$$

This is a two-dimensional birth death process on the state space $E := \mathbb{N}_0 \times \mathbb{N}_0$ so that the chain has 6 possible jumps with rates

$$
c((i,k),(j,l)) = \begin{cases} \frac{1}{L-1}(N-i-k)(d+i) & \text{if} \quad j = i+1, l = k \\ i\left(d + \frac{N-i-k}{L-1}\right) & \text{if} \quad j = i-1, l = k \\ \frac{1}{L-1}(N-i-k)(d+k) & \text{if} \quad l = k+1, i = j \\ k\left(d + \frac{N-i-k}{L-1}\right) & \text{if} \quad l = k-1, i = j \\ \frac{1}{L-1}k(d+i) & \text{if} \quad j = i+1, l = k-1 \\ \frac{1}{L-1}i(d+k) & \text{if} \quad j = i-1, l = k+1. \end{cases} \tag{5.39}
$$

It can be seen that rates vanish as $L \to \infty$ for the interacting terms (last two terms), so $\eta_x(t)$ and $\eta_y(t)$ become independent birth death chains with rates as in (5.2). The same will be true for any fixed number of sites.

## 5.7 Explosive condensation processes and gelation

We discuss another processes of type (2.24) that exhibits condensation and has attracted significant recent research interest. Even though it is not covered by Theorem 3.1 or Theorem 5.1, we include it to illustrate the possible irregular behaviour and non-existence of solutions to (3.9) related to gelation in growth/aggregation models. Explosive condensation processes (ECP) have been introduced in [95] and further studied in [35, 21] on a heuristic level. The jump rates are of the form

$$c(k,l) = k^\lambda(d + l^\lambda) \quad \text{with parameters } \lambda \geq 1 \text{ and } d \geq 0, \tag{5.40}$$

are unbounded and diverge super-linearly with occupation numbers on departure and target sites. For $\lambda = 1$, this model is the inclusion process which we discussed previously. While our results do not apply to the case $\lambda > 1$, (3.9) still represents the only possible limit dynamics for $f_k(t)$, and we expect convergence to actually hold at least as long as it has a unique solution. Rates of the form (5.40) are related to collision kernels in aggregation models which have attracted significant research interest (see e.g. [21] and references therein).

The rates (5.40) for $\lambda > 1$ satisfy condition (2.27) and we have product measures of the form (2.30) with $\phi_c = 1$ and

$$w(n) = \prod_{k=1}^{n} \frac{(k-1)^\lambda + d}{k^\lambda} \sim n^{-\lambda}, \quad \text{as } n \to \infty. \tag{5.41}$$

Therefore, $\rho_c < \infty$ for $\lambda > 2$ and as for models with bounded rates we expect $f(t) \to f^\phi$ as $t \to \infty$ for all initial conditions with $m_1(0) = \rho \leq \rho_c$. If $\rho > \rho_c$, we expect a scaling solution in analogy to zero-range processes as discussed in Section 4.2.2.

The exchange-driven growth model studied in [12] corresponds to rates (5.40) in the degenerate case $d = 0$, and provides a detailed analysis of the condensed part of the scaling solution. Note that in this case $w(n) = \delta_{0,n}$ and the mean-field equation has an absorbing state corresponding to $f_k = \delta_{0,k}$ as the only stationary distribution for all $\lambda > 0$, effectively setting $\rho_c = 0$. Still, $m_1(t)$ is conserved and the dynamics of the particle system is not irreducible, more and more sites become empty over time and cannot get occupied again thereafter. The limiting master equation (3.9) can be written as

$$\frac{d}{dt} f_k(t) = m_\lambda(t) \big[ (k+1)^\lambda f_{k+1}(t) + (k-1)^\lambda f_{k-1}(t) - 2k^\lambda f_k(t) \big], \tag{5.42}$$

for all $k \geq 0$, using $f_{-1}(t) \equiv 0$. This involves the moment $m_\lambda(t)$ which can be absorbed in a time change $\tau_t = \int_0^t dt' m_\lambda(t')$, leading to a standard birth death chain with symmetric rates $k^\lambda$. Note that for $\lambda = 1$ and $m_1(t) = \rho$ this corresponds to (5.3) for the inclusion process. Since $\rho_c = 0$ all initial conditions with $\rho = m_1(0) > 0$ lead to phase separated solutions of the form (4.6)

$$f_k(t) = f_k^{\mathrm{bulk}}(t) + f_k^{\mathrm{cond}}(t), \tag{5.43}$$

with $f_k^{\mathrm{bulk}}(t) \to \delta_{k,0}$ as for the inclusion process with $d = 0$. The results reported in [12] refer to $f_k^{\mathrm{cond}}$, which for $\lambda < 2$ again exhibits a scaling form

$$f_k^{\mathrm{cond}}(t) = \epsilon_t^2 h(u), \quad \text{with} \quad u = k\epsilon_t \ , \ \epsilon_t = \tau_t^{-\alpha} \text{ and } \alpha = \frac{1}{2-\lambda}. \tag{5.44}$$

The scaling function again satisfies a second-order linear differential equation

$$(2-\lambda) \frac{d^2}{du^2} (u^\lambda h(u)) + u \frac{d}{du} h(u) + 2h(u) = 0, \tag{5.45}$$

subject to normalization, which has an explicit solution

$$h(u) = \frac{(2-\lambda)^{2/(2-\lambda)}}{\Gamma(1/2-\lambda)} u^{1-\lambda} \exp\left( -\frac{x^{2-\lambda}}{(2-\lambda)^2} \right). \tag{5.46}$$

For $\lambda > 2$ there is no solution to the limit dynamics (3.9), which exhibits instantaneous blow up of second moments – also called gelation in the context of aggregation models (see e.g. [8]). On the level of the particle system this corresponds to the explosive condensation phenomenon studied in [95, 35, 21] for $d > 0$, where the time to reach the condensed state vanishes with increasing system size even in one-dimensional geometries. On the complete graph with $d = 0$ the behaviour can again be characterised through the second moment as reported in [12],

$$m_2(t) \sim \begin{cases} t^\beta & , \ \lambda < 3/2 \\ \exp(Ct) & , \ \lambda = 3/2 \quad \text{for some } C > 0 \\ (t_c - t)^\beta & , \ 3/2 < \lambda < 2 \quad \text{for some } t_c > 0 \\ \infty & , \ \lambda > 2 \end{cases} . \tag{5.47}$$

The dynamical exponent for the power law cases above is given by $\beta = (3 - 2\lambda)^{-1}$, and for $\lambda > 3/2$ the system exhibits finite-time blow up at $t_c$, which becomes instantaneous for $\lambda > 2$. The boundary case $\lambda = 2$ shows interesting multiscaling behaviour as discussed in [12] Section 3B. Note that for $d > 0$ with (5.41) only $\lambda > 2$ leads to $\rho_c < \infty$ and condensation is always explosive as mentioned above.

However, the example of explosive processes with $\lambda > 2$ shows that some growth conditions on the rates are necessary for convergence to (3.9) to hold. In cases of instantaneous blow up, the single site process $\eta_x(t)$ does not have well-defined limit dynamics for any $t > 0$.

## 5.8 Conclusion

We show that the single site dynamics of the inclusion process is a birth death chain even for fixed system size. The limiting process is also a birth death chain, described by a mean-field equation without non-linearity. Since the master equation of the birth death chain (5.3) is simple, it can be analysed via the common generating function approach where we can get the mean and variance of the process. We also study the coarsening dynamics towards condensation for the inclusion process with $d = 0$ via the site and size-biased empirical processes as we did in Chapter 4. The use of the size-biased birth death chain for the inclusion process still provides a strong tool to study the coarsening dynamics without absorption and significantly improved statistics. We get a simple linear birth death chain, the rates of which depend only on $\rho$ unlike what we had before for the zero-range process. For the case $d > 0$, there is no condensation without scaling $d = d_L \to 0$ with the system size. This can be studied using duality, which provides a powerful tool for all values of $d$ and system sizes for exact computations for the second moment of the process. The calculation for the complete graph case is rather straightforward and could also be applied to the 3-particle-dual system to compute the third moment or used on other geometries such as regular lattices with periodic boundary conditions and symmetric dynamics. The intermediate case $dL \to \alpha \in (0, \infty)$ leads to hierarchical configuration with several macroscopic clusters related to Poisson-Dirichlet distributions, which is currently under investigation. These distributions have been observed before e.g. in spatial random permutations related to Bose-Einstein condensation. Lastly, we expect convergence to the limiting mean-field master equation could also apply to a more general case such as the explosive condensation process, which has been studied in the literature for the case $d = 0$ as discussed in Section 5.7.

# Chapter 6

# Conclusion and Outlook

In this thesis, we studied several interacting particle models of misanthrope type that exhibit a condensation transition. We focused on models that are ergodic on finite systems with homogeneous initial conditions on a complete graph, and in particular, have factorised stationary measures. We first established a rigorous derivation of the mean-field equation for stochastic particle systems of misanthrope type with bounded rates. This single site mean-field master equation helps to study the coarsening dynamics related to condensation. We also introduced a new size-biased version of the single site dynamics which provides an effective tool to analyse the dynamics of the condensed phase without finite size effects.

In Chapter 3, we introduced the main mean-field master equations and presented some rigorous results that are used throughout the thesis. We considered the single-site dynamics in stochastic particle systems of misanthrope type with bounded rates on a complete graph. In the limit of diverging system size, we establish convergence to a Markovian non-linear birth death chain, described by a mean-field equation also known from exchange-driven growth processes. Conservation of mass in the particle system leads to conservation of the first moment for the limit dynamics, and to non-uniqueness of stationary measures. The proof we provided is based on a coupling to branching processes via the graphical construction. Our result provides a contribution towards a rigorous understanding of the connection of the mean-field equations to underlying particle systems and of coarsening dynamics in condensing systems.

In Chapter 4, we considered a zero-range process for which the jump rates are bounded so that we could use the mean-field master equation derived in Chapter 3. The jump rates which decrease with occupation number are chosen so that the system exhibits a condensation transition. The time evolution of the condensed phase exhibits an interesting coarsening phenomenon of mass transport between cluster sites characterised by a power law. The derived single site master equation is used to describe the coarsening behaviour. We revisited and extended the approach in [44] to a larger class of parameter values which gives the coarsening scaling law depending on system parameters. We establish a new method based on the size-biased process, which provides an effective tool to sample the dynamics of the condensed phase without finite size effects and with significantly improved statistics, which is the main novelty of this chapter.

In Chapter 5, we studied the inclusion process, that under certain conditions also exhibits a condensation transition. For this process, our technique of deriving the mean-field master equation in Chapter 3 does not hold since the jump rate is not bounded. However, it turns out that for the inclusion process, the single site dynamics are a Markovian birth death chains even for finite system size. The limiting process is also a birth death chain, described by a mean-field equation but without non-linearity which simplifies due to mass conservation. The use of the size-biased process for the inclusion process still works well and provides a strong tool to study the coarsening dynamics for a special case of the jump rates. We also derived some exact results on the system through duality. We gave exact computations of the time dependent covariance using the self-duality of inclusion processes and a two-particle dual process. By considering this covariance, we were able to generalise previous results on the coarsening regime.

Several interesting and important open questions follow directly from work in this thesis. Firstly, it would be interesting to establish the mean-field equation for a more general case. We expect the limit result to hold for more general conditions on jump rates as long as the system does not exhibit the explosive condensation, but this would require a significant extension of our proof. The i.i.d initial conditions could also be relaxed to a more general condition as discussed in Chapter 3. Since we know that mean-field equations provide an approximation for other geometries, one possible extension is to apply our argument to graphs which are not complete. Although the derivation might not be rigorous, the approach of a size-biased process is generic and can be adapted to other condensing particle systems such as explosive condensation processes. For the inclusion process, we could also consider a finite system which follows a birth death master equation. Also, the case with $d > 0$ could be interesting if we scale $d \to 0$ to see the coarsening behaviour deriving from the birth death chain. The analysis with duality also provides insights into the understanding of the inclusion process, and it posed significant challenges to give rigorous proofs of the dynamics, in particular for asymmetric systems. It would also be interesting to explore potential applications of this work further, for example, the zero-range process as a traffic model (on roads or other networks), or the inclusion process as a simple model for evolutionary dynamics in biological systems or wealth condensation in macroeconomics.

# Appendix A

# Propagation of Chaos

The propagation of chaos is a concept originating from kinetic theory, where it serves to relate the equations of Boltzmann and Vlasov to the dynamics of many-particle systems (see [88] and references therein). Propagation of chaos means that stochastic independence of random quantities under the initial distribution is preserved in time under the dynamics. For interacting particle systems, this usually only holds in the limit of diverging system size.

Propagation of chaos for symmetric simple exclusion processes is studied in [87]. Assuming that particles are initially located independently on the lattice, it is shown that observing a finite number of different particles leads to independent diffusions in a hydrodynamic scaling limit. The interactions on the microscopic scale vanish on large space and time scales due to the hydrodynamic scaling. This is also related to the dynamics of tagged particles which have been studied in great detail for exclusion processes and related models, as explained in Section IV.3 in [68].

Another origin of the propagation of chaos is the presence of mean-field interactions, which lead to self-averaging of local dynamics in the limit of large system size even without time rescaling. This has been studied e.g. in [27] for systems of birth death processes with mean-field interaction to model a class of chemical reactions. This result covers a law of large numbers as a deterministic dynamical system and a central limit theorem for the asymptotic fluctuations. In our result presented in Chapter 3 we cover the case of mean-field interactions in misanthrope processes, which has not been addressed so far to our knowledge.

Following the lecture notes [84] (where this is worked out in detail for mean-field interacting diffusions), we will make use of the following equivalent characterization of the chaos property formulated in a general context.

**Proposition A.1.** *Let* $(X^{i,N} : 1 \leq i \leq N, N \geq 1)$ *be a triangular array of random* *variables taking values in a topological space $E$, such that for each $N$, the law of $(X^{i,N})_{1\leq i\leq N}$ is symmetric, i.e. invariant by permutation of components. Moreover, let $(\bar{X}^i)_{i\geq 1}$ be an i.i.d. sequence of $E$-valued random variables. Then the following statements are equivalent:*

*(a) for all $m \geq 1$,*

$$(X^{1,N}, X^{2,N}, ..., X^{m,N}) \to (\bar{X}^1, \bar{X}^2, ..., \bar{X}^m)$$

*in distribution as $N \to \infty$,*

*(b) the sequence of empirical measures on $E$*

$$p_N(dy) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}}(dy)$$

*converges in distribution to $Q := \mathrm{Law}(\bar{X}^1)$ as $N \to \infty$.*

In Chapter 3 we show that under independent initial conditions, (b) holds for all $t \geq 0$ for empirical measures of particle configurations on the state space $E$, establishing the propagation of chaos.

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